R. A. Daly

Keep! Important

R. A. Daly

DEPARTMENT OF THE INTERIOR

BULLETIN

OF THE

UNITED STATES

GEOLOGICAL SURVEY

No. 48

WASHINGTON
GOVERNMENT PRINTING OFFICE
1888

UNITED STATES GEOLOGICAL SURVEY

J. W. POWELL, DIRECTOR

ON THE FORM AND POSITION

OF

THE SEA LEVEL

WITH SPECIAL REFERENCE TO ITS DEPENDENCE ON SUPERFICIAL MASSES SYMMETRICALLY DISPOSED ABOUT A NORMAL TO THE EARTH'S SURFACE

BY

ROBERT SIMPSON WOODWARD

WASHINGTON
GOVERNMENT PRINTING OFFICE
1888

CONTENTS.

KEY TO MATHEMATICAL SYMBOLS.

[**The page numbers refer to pages where the symbols are first introduced and explained in the course of the investigation.**]

		Page.
a	=a special value of the radius-vector of the sea surface used in the investigations of Archdeacon Pratt	71
a_0	=the equatorial semi-axis of the earth's spheroid or that oblate spheroid which coincides closely with the sea surface	19
B	=a certain definite integral	55
b	$=\sin \frac{1}{2}\beta$	43
b_0	=polar semi-axis of the earth's spheroid	19
b_0	=the greatest value of b when it is variable	55
C_1	=a constant used temporarily for brevity	19
C_2	=a constant used temporarily for brevity	20
c	$=2\, r_0 \sin \beta$	50
$c, d, e \ldots$	=symbols for constant coefficients used only temporarily	44
c_0	=a constant used temporarily for brevity	36
D	=the distance between the attracted and any attracting point.	21
$F_i(\beta)$	=a definite integral, a function of the angle β, i being any positive integer	33
$f_i(\cos \theta)=f_i(\mu)$	=a polar harmonic, a function of the angle β, i being any positive integer	33
g	=the velocity increment due to the earth's attraction on bodies outside of and near to its surface; g is about 32 feet if the mean solar second is the unit of time	20
$g_1, g_2, g_3 \ldots$	=symbols for functions of b or the angle β	44
h	=the thickness of the attracting mass measured along a radius of the earth	22
h_0	=the maximum value of h when it is variable	53
h_1, h_2	=height above sea level to which a stratum is supposed to be raised	83
I, I_1, I_2	=certain definite integrals, I being used for either I_1 or I_2 when it is not necessary to particularize	24
i	=any positive integer	32
$J_0,\ J_1,\ J_2 \ldots$	=symbols for certain definite integrals	32
$J_0',\ J_1',\ J_2' \ldots$	=symbols for certain definite integrals	32
$J_0'', J_1'', J_2'' \ldots$	=symbols for certain definite integrals	32
$j_1, j_2, j_3 \ldots$	=constants which are functions of the angle α.	55
$k_1, k_2, k_3 \ldots$	=constants which are functions of the angle β	45
l	=the distance of any point of the sea level from the earth's axis of rotation	18
$\log_e$	=Naperian logarithm	32
M	=the mass of the earth	19
m	=a disturbing or attracting mass situated anywhere with respect to the earth, and especially on or near the earth's surface	19
$N_1, N_2, N_3 \ldots$	=symbols for certain numerical quantities	65
n	=any positive integer	44

Symbol	Definition	Page.
P	=the potential of the earth's mass with respect to a particle or unit mass on its surface. P is the sum of every mass-element of the earth divided by its distance from the attracted particle	18
P_0, P_1, P_2 . . .	=symbols for spherical harmonics or Laplace's coefficients	31
p	=an auxiliary angle and subject variable of a definite integral	23
Q_1, Q_2	=certain factors used temporarily for brevity	29
q	=an auxiliary angle	23
r	=the radius-vector of any element of the attracting mass	21
r_0	=the radius of a sphere of equal volume with the earth's spheroid. It is the radius of the sphere of reference	19
r'	=the radius-vector of the attracted point or particle	21
S, S_1	=sums of certain series	56
s, s_0	=auxiliary angles used temporarily	23
t, t_1, t_2	=auxiliary quantities, t being the subject of a definite integration, and t_1 and t_2 the limits of t	38, 39
t_1, t_2	=thicknesses of strata	83, 84
U_0	=a constant used temporarily	36
u	=the excess of r over r_0	21
V	=the potential of the disturbing mass m	19
V_0	=a constant. It is the value of V along the line of intersection of the disturbed and undisturbed surfaces	20
v	=the elevation or depression of the disturbed sea surface relative to the undisturbed sea surface. It is the same as $(r'-r_0)$	20
v_0	=a constant used temporarily	20
v_1, v_2, v_3	=certain special values of v	41
v'	=the elevation or depression of the sea surface relative to a spherical surface concentric with the disturbed center of gravity of the earth	52
v''	=elevation or depression of the disturbed sea surface relative to the undisturbed surface when the disturbing mass is of variable thickness	53
v_1'', v_2'', v_3''	=certain special values of v''	56
v_a	=elevation of disturbed surface at a point whose angular distance from the center of the disturbing mass is α	70
w	$=\sin \frac{1}{2}a / \sin \frac{1}{2}\beta$	43
X	=a definite integral	46
$x, y, z,$	=rectangular Cartesian co-ordinates	21
x', y', z'	=rectangular Cartesian co-ordinates	21
$Y_0, Y_1, Y_2,$ . . .	=certain spherical harmonics or Laplace's coefficients	36
$Z_0, Z_1, Z_2,$ . . .	=certain spherical harmonics or Laplace's coefficients	36
α	=the angular distance of any point of the disturbed sea surface from the center of the disturbing mass. It is the angle at the center of the sphere of reference between a line drawn to any point of the disturbed surface and a line drawn to the center of the disturbing mass	23
β	=the angular radius of the disturbing mass or the angle at the center of the sphere of reference between a line drawn to the center of the disturbing mass and one drawn to its border. β is used in this sense with reference to masses of uniform thickness	23
β_0	=the greatest value of β when it is variable; i. e., β_0 is the angular radius of a mass whose thickness is variable	53
β_1, β_2	=limits of β when it is a subject of integration	53.

		Page.
β'	=the angular radius of a conical ring or annulus and a subject of integration	51
γ_1, γ_2	=auxiliary angles and subject variables in the definite integrals I_1 and I_2	24
ΔV	=a finite change in or increment to V	36
ΔV_0	=a finite change in or increment to V_0	36
Δv	=a finite change in or increment to v	36
$\Delta \rho$	=a finite change in or increment to ρ	58
η	=an ordinate and subject variable	49
θ, θ'	=angles corresponding to polar distances	21
λ, λ'	=angles corresponding to longitudes	21
ν	=cosec $\frac{1}{2}\alpha$	45
ξ	=an ordinate and subject variable	49
π	=the ratio of the diameter to the circumference of a circle =3.14159+	19
ρ	=the density of the attracting or disturbing mass m	19
ρ	=the mean density of the earth=5.5, as used in this paper	19
ρ_w	=the density of sea water=1, as used in this paper	36
ρ_1, ρ_2	=the densities of strata whose thicknesses are t_1 and t_2, and heights above sea level h_1 and h_2	83, 84
σ	=the displacement of the earth's center of gravity, caused by a superficial mass having a circular border	51
σ_1	=the displacement of the earth's center of gravity due to the shifting of a hemispherical meniscus from one hemisphere to the opposite one	78
τ	=the thickness of a spherical shell of equal volume with assumed ice mass	64
$\varphi(\beta)$	=the variable thickness of an attracting or disturbing mass, β being the angular distance of any part of the mass from its center	53
ψ	=an auxiliary angle used temporarily only	21
ω	=the angular velocity of the earth about its axis	18
ω	=the ratio of the area of the ocean to the area of the earth's surface in a case assumed by Sir W. Thomson	76

LETTER OF TRANSMITTAL.

DEPARTMENT OF THE INTERIOR,
U. S. GEOLOGICAL SURVEY,
Washington, D. C., May 31, 1887.

SIR: I have the honor to transmit herewith the results of certain investigations, which may be broadly designated as relating to the form and position of the sea level. These investigations were begun in part previous to my connection with the Geological Survey, but they were taken up again in 1885, with your approval, at the request of Mr. G. K. Gilbert and Prof. T. C. Chamberlin, for solutions of some special problems which arose in their geological researches. The work has been prosecuted simultaneously with other lines of office and field work. It reached its present form substantially, however, more than a year ago; and the principal numerical results of the discussion of Professor Chamberlin's problem are incorporated with his paper on The Driftless Area, in the Sixth Annual Report. The purely mathematical features of the paper have been published also in the Annals of Mathematics, Nos. 5 and 6, vol. 2, and No. 1, vol. 3. I have delayed offering the complete manuscript for publication up to this time in order that I might give it a careful revision and check all the more important formulas by independent processes of derivation.

The questions treated in this paper are for the most part necessarily somewhat mathematical. They are, however, fundamental questions in geophysics, and although the mathematical form of presentation has been followed throughout, an attempt has been made to state the end results and formulas in such a way that they may be understood and used with safety by those who may not care to follow the details of the analysis. For the benefit of such readers a key to the mathematical symbols employed is given in addition to the list of contents and general index.

While the analysis of this investigation was designed especially to solve the particular problems of Messrs. Gilbert and Chamberlin, it has not been confined to those problems, but has been adapted to the entire class of problems to which they belong. It is hoped, therefore, that the results of the paper will be of interest and value to geodesists and mathematicians as well as to geologists.

Very respectfully, your obedient servant,

R. S. WOODWARD.

Hon. J. W. POWELL,
Director U. S. Geological Survey.

ON THE FORM AND POSITION OF THE SEA LEVEL.

By R. S. Woodward.

I. INTRODUCTION.

1. The problem of the form and dimensions of the sea level surface of the earth has been one of peculiar difficulty. The combined efforts of the ablest mathematicians of the past two centuries, supplemented by the most laborious and costly geodetic measurements have yielded us the first approximation only to the complete solution. Fortunately this first approximation is exceedingly close. It assigns to the sea level a form which differs but slightly from that of an oblate spheroid, whose major and minor semi-axes are about 20,926,000 and 20,855,000 English feet, respectively. This spheroid, or reference ellipsoid, as it is sometimes called, has its minor axis coincident with the earth's axis of rotation and is usually regarded as sensibly fixed in position and dimensions. With respect to it the actual sea surface or geoid must be imagined to lie partly above and partly below by small but unknown amounts, the determination of which, if possible, will constitute a second approximation to the figure of the earth. For many if not most of the applications of science the reference ellipsoid suffices; the first approximation is nearly enough correct. But geodesy, on the one hand, has attained such a degree of perfection in precise measurement that the discrepancies now brought to light in some of its operations must be attributed largely if not chiefly to defects in theory. These discrepancies must be explained before any considerable advance can be expected in our knowledge of the figure of the earth along the present lines of investigation. Their true explanation is apparently intimately connected with the form of the geoid, and it is to the study of the form, therefore, rather than to the determination of the dimensions of the geoid that we may look for future progress in geodesy. Geology, on the other hand, has raised many questions relative not only to the form, position, and fixity of the geoid proper, but also with respect to the allied equipoten-

tial surfaces of isolated bodies of water at higher or lower levels. It is found, for example, in geological investigations, that the shore lines of extinct seas do not always coincide with existing level lines, but often cross them at decided angles, or that the water level lines traced on islands in such extinct seas differ in elevation from contemporaneous lines traced on their distant shores. Aside from the changes which may have been due in these cases to subsidence or upheaval, the question may be raised whether such slopes or differences in elevation relative to present level lines may not have been caused by adjacent attracting masses, which have since disappeared, like the ice mass of the glacial epoch, or in a lake basin by the presence of the water itself. Correct and complete answers to such questions require a knowledge of the existing geoid and of the causes which may have produced secular variations in its form and position.

At present it is by no means clear how any specially extensive additions to our information concerning the more minute features of the sea surface are to be obtained. It may even be doubted whether we have not reached a practical limit in the first approximation; whether in fact the distribution of matter within the earth's surface is not so irregular as to preclude gaining anything more than an empirical formula for the deviations of the geoid from the ellipsoid of reference. It seems probable, however, that the forces producing these deviations have their seat in a comparatively thin terrestrial crust resting on a fluid or plastic substratum (or nucleus), or that such was the antecedent condition of the earth, and that our failure to perceive the relations of the crust to the substratum is the chief obstacle to improvement.

In the absence of a complete rational theory the best evidence which analysis can bring to bear on questions pertaining to the geoid is largely of a negative character. The effects which would result under certain conditions can be computed, but it is not always possible to prove that those conditions accord with the actual facts. Investigation must proceed to some extent upon doubtful postulates, and computations must be made from uncertain data. But notwithstanding this limitation on the calculations we are about to consider, they will generally possess a value in excluding or confirming hypotheses, or in furnishing limiting values for the effects of observed causes.

2. A considerable class of problems concerning the sea level is that in which the attracting or disturbing mass is symmetrically disposed about a radius of the earth's surface, and is situated on or near the surface. As examples of this class we may adduce the two following, which led to this investigation:

(*a*) Given the dimensions of a lake basin having a circular border. When the lake was full of water it left a trace of its surface along the border and on an island at the center of the basin. After the water had disappeared a line of spirit levels was run between the water trace

on the island and that on the border; what difference in altitude should have been found?[1]

(*b*) Assuming the accumulation of ice in glacial times to have been in the shape of a spherical stratum bounded by a circle, or some sort of meniscus symmetrical about an axis, and that the earth's crust did not yield under the weight of the ice, what were the resulting distortions in the sea level?[2]

It will be seen that these problems are essentially the same. They are substantially identical also with the problem of the effect of continental masses on the sea level, since the continents may be represented, approximately at least, as spherical strata having circular borders, or as masses of meniscoid shape.

3. The following paper is devoted to the investigation and discussion of this class of problems. An attempt has been made to develop the theory of their solution so far as is necessary to render practicable the numerical evaluation of the characteristic effects of the disturbing mass in any special case. In Articles II to XI the theory of the effect of a mass in the shape of a spherical stratum having a circular border and uniform thickness is worked out with considerable detail. The only restrictions imposed on this mass are that its density is uniform, and that the ratio of its thickness to the earth's radius may be neglected in comparison with unity. Expressions for the potential of the disturbing mass at any point of the disturbed surface are derived in terms of a definite integral and in terms of spherical harmonics; and the degree of approximation of these expressions is investigated. Equations to the disturbed surface are assigned for the case in which the effect of the rearranged free water is considered, as well as for the case in which that effect is neglected. The disturbance in the former case is shown to be equal to that in the latter, which is expressed in compact integral form, plus a rapidly converging series of additive terms.

---

[1] This problem was proposed by my colleague, Mr. G. K. Gilbert, to the mathematical section of the Washington Philosophical Society, February, 1884. In his geological investigations within the area of the Quaternary sea known as Lake Bonneville, Mr. Gilbert has found traces of the central portions of the ancient lake surface to be more than 100 feet higher than the traces of the contemporaneous surface at its margin. A complete consideration of the effects of the causes which might contribute to this distortion requires, obviously, a numerical evaluation of the depression of the level surfaces within the area, due to the removal of the water.

A solution of the problem was given by the writer before the above-named society in March, 1884, and a more complete discussion will be found in sections **39–41**.

[2] To what extent the form and position of the sea level may be modified by the mere attraction of glacial masses is a question which has been much discussed by geologists. It was proposed to the writer by Prof. T. C. Chamberlin, geologist in charge of the division of glacial geology, U. S. Geological Survey. The question is considered at some length in sections **42–52**, and a review of the work of the more prominent mathematicians who have discussed the problem is given in sections **53–62**. The principal numerical results of the writer's investigations are given in Professor Chamberlin's paper on The Driftless Area, in the Sixth Annual Report of the U. S. Geological Survey.

In Article XII the investigation is extended so as to assign the effect of any mass of uniform density having a symmetrical distribution about a radius of the earth's surface. Particular attention is paid to a class of masses whose shapes are assigned by a formula which represents fairly well the mass features of the problems (*a*) and (*b*) above.

Under the head of applications, Articles XIII to XVI, the characteristic properties of the equipotential surfaces in a lake basin are first considered. Then the variations in sea level attributable to continental glaciers or ice caps are discussed at some length. The angular radial extent of the ice mass is, for the most of the discussion, assumed to be 38°, for the reason that this is the extent of a mass of nearly uniform thickness, which would produce the maximum upheaval of the water along its border. The external shapes of the various masses, their volumes, and the distortions of the sea surface attributable to them are given in detail. The minimum thicknesses of ice masses of varying angular radial extent, requisite to produce average slopes of five feet per mile within one degree (69 miles) of their borders, and the extent of variation in sea level on the hypothesis of an alternation of glaciation at the poles of the earth, are also worked out.

In the historical note of Article XV, the allied investigations of Archdeacon Pratt, Mr. D. D. Heath, and Sir William Thomson, on the problem of glacial submergence are reviewed. The special cases they have considered are shown to be easily derived from the general formula of Article XII.

Finally, in Article XVI, a brief discussion of the effect of continental masses in distorting the sea level is given. It is shown that according as the continents are or are not superficial masses unbalanced in their attractive effects, the sea surface must be very irregular or deviate only by minute quantities from the ellipsoidal form. It is also shown that although a continent whose radial element masses are in a condition bordering on hydrostatic equilibrium would produce but slight disturbances in the position of the sea level, it might nevertheless cause a considerable slope of the sea surface, or deflection of the plumb line along its border.

A. THEORY.

II. MATHEMATICAL STATEMENT OF PROBLEM.

4. The solution of the general problem outlined in the preceding section depends on the principle of hydrostatics that the potential of the forces producing a liquid surface in equilibrium has a constant value for all points of that surface. In the case of the earth, if the potential of all the attractive forces acting on a unit mass at any point of the sea surface be denoted by P, the distance of the point from the earth's axis of rotation by l, and the velocity of rotation by ω, the form of the surface will be completely defined by the equation

$$P+\tfrac{1}{2}l^2\omega^2=\text{a constant.} \tag{1}$$

The exact value of P in this equation is a complicated function of the densities of the element particles of the earth and of the co-ordinates of those particles and the attracted point. For the present purposes, however, it will be sufficient to consider P due to a centrobaric sphere of equal mass and volume with the earth and concentric with the earth's center of gravity. Since we shall only consider relative positions of any point on the sea surface, the potential due to centrifugal force, which is represented by the second term in (1), may be neglected.

5. If a_0 and b_0 denote the equatorial and polar semi-axes, respectively, of the earth's ellipsoid, and r_0 the radius of the sphere just referred to,

$$r_0=\sqrt[3]{a_0^2b_0}. \tag{2}$$

Using Clarke's values[1] of a_0 and b_0 we have

$$\begin{aligned} a_0&=20926062 \text{ English feet,}\\ b_0&=20855121 \text{ English feet,}\\ r_0&=20902394 \text{ English feet,}\\ \log r_0&=7.32020. \end{aligned}$$

The surface of the sphere thus defined may be regarded as the surface assumed by a thin film of sea water covering a nucleus whose mass, plus the mass of the film, equals the earth's mass. We shall call this ideal surface the undisturbed surface. With respect to it the real surface of the earth lies partly without and partly within; but so far as small relative changes in sea level are concerned it is practically immaterial whether we refer to the actual closely spheroidal surface or to the simpler spherical one.

6. Let

$$\begin{aligned} M&=\text{mass of the earth,}\\ \rho_m&=\text{mean density of earth.} \end{aligned}$$

Then,

$$M=\tfrac{4}{3}\pi r_0^3\rho_m, \tag{3}$$

and the equation to the undisturbed surface is

$$\frac{M}{r_0}=\frac{4}{3}\pi r_0^2\rho_m=C_1, \tag{4}$$

C_1 being a constant.

Suppose, now, a new mass, m, of density ρ (positive or negative) be placed in any fixed position relatively to the undisturbed surface. The resulting sea surface will then differ from that defined by (4). To determine this difference let V be the potential of the disturbing mass m

[1] Comparisons of Standards of Length, made at the Ordnance Survey Office, Southampton, England, by Capt. A. R. Clarke, R. E. Published by order of the secretary of state for war, 1866.

at any point of the disturbed surface, and let v denote the elevation or depression of this point relative to the undisturbed surface. The equation to the disturbed surface will then be

$$\frac{M}{r_0+v}+V=C_2,\text{ a constant.} \tag{5}$$

The difference of this and (4) to terms of the first order inclusive in v is

$$-\frac{M}{r_0^2}v+V=C_2-C_1,$$

whence, putting

$$V_0=C_2-C_1,$$

$$v=(V-V_0)\frac{r_0^2}{M}. \tag{6}$$

Since $M/r_0^2=g$, the velocity increment at the earth's surface due to the earth's attraction, (6) may be written

$$v=\frac{V-V_0}{g}. \tag{6)'$$

V_0 in the last two equations is the value of V when $v=0$, or the value of V along the line of intersection of the disturbed and undisturbed surfaces. If we put

$$v_0=V_0\frac{r_0^2}{M},$$

$$v+v_0=V\frac{r_0^2}{M}=\frac{V}{g}. \tag{7}$$

This equation represents the elevation of the disturbed surface above a spherical surface of equal potential, whose value is

$$\frac{M}{r_0-v_0}=C_2,$$

since the difference between this and (5) gives (7).

The constant V_0 may be determined from the obvious condition that the disturbed and undisturbed surfaces must contain equal volumes.

It is evident that the equations just derived will hold true if the mass m be a part of the earth's mass, so long as the ratio m/M may be neglected relatively to unity. Thus, in the problems we shall consider, m may represent the mass of a continent, the deficiency in mass of a lake or lake basin, or the ice mass of the glacial epoch.

III. EVALUATION OF POTENTIAL V—DISTURBING MASS ON SURFACE OF EARTH, OF UNIFORM THICKNESS AND DENSITY, AND WITH CIRCULAR BORDER.

7. The next step in the solution requires the determination of the potential V of the attracting mass for any point of the disturbed surface, whether without or within the circle which we have assumed to define the boundary of the mass. Although the nature of the mass may be such as to prevent the water from permeating it freely, the surface the water would take if not so restricted is an essential part of the disturbed surface.

In order to derive an expression for V, let the rectangular and polar co-ordinates of any point of the attracting mass be defined by the usual relations, viz:

$$x=r\cos\theta\cos\lambda,$$
$$y=r\cos\theta\sin\lambda,$$
$$z=r\sin\theta,$$

in which θ and λ correspond to polar distance and longitude, respectively, the position of the origin being arbitrary. With reference to the same origin, let the co-ordinates of the attracted point on the sea surface be

$$x'=r'\cos\theta'\cos\lambda',$$
$$y'=r'\cos\theta'\sin\lambda',$$
$$z'=r'\sin\theta'.$$

If D denote the distance between the attracting and attracted points and

$$\cos\psi=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\lambda-\lambda'), \tag{8}$$

$$D^2=r^2+r'^2-2rr'\cos\psi=(r-r')^2+4rr'\sin^2\frac{\psi}{2}. \tag{9}$$

The volume element of the attracting mass is

$$dxdydz=r^2dr\sin\theta d\theta d\lambda.$$

Hence, if ρ denote the density of the attracting mass, a general expression for the required potential is

$$V=\rho\int\int\int\frac{r^2dr\sin\theta d\theta d\lambda}{D}. \tag{10}$$

We must now evaluate this integral. Taking the center of the sphere of reference as the origin of co-ordinates, let

$$r=r_0+u,$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (11)

$$r'=r_0+v,$$

in which u and v are small quantities relative to r_0, v being the same of course as defined by equation (6). Premising what will be proved in Article IV, namely, that we may neglect quantities of the order $\frac{u}{r_0}$, $\frac{v}{r_0}$, and $\left(\frac{u-v}{r_0}\right)^2$, equation (9) gives

$$D=2r_0 \sin\frac{\psi}{2}. \tag{12}$$

From the first of equations (11)

$$\left.\begin{aligned} dr&=du, \\ \text{and} \qquad r^2&=r_0^2, \end{aligned}\right\} \tag{13}$$

to terms of the order $\frac{u}{r_0}$.

As to the magnitude of the quantities neglected, it may be remarked in passing that r_0 is in round numbers 21,000,000 feet (see section 5), while u and v may be restricted to values less than 100,000 feet; so that the fractions neglected will not exceed $\frac{1}{200}$.

Without loss of generality we may assume the line from which θ and θ' are reckoned to pass through the attracted point, and the plane from which λ and λ' are reckoned to pass through the attracted point and the center of the attracting mass. In this case $\theta'=0$ and $\lambda'=0$, and (8) gives $\psi=\theta$.

By means of this relation and the equivalents in (12) and (13) the integral in (10) becomes

$$V=r_0\rho\int\int\int du \cos\frac{\theta}{2}\, d\theta d\lambda. \tag{14}$$

If the uniform thickness of the attracting mass be denoted by h, the limits of u in (14) will be 0 and h. Let the limits of θ, which are obviously functions of λ, be denoted by θ_1 and θ_2. The limits of λ are evidently equal in magnitude but of opposite signs. Hence we have

$$V=2r_0\rho\int_0^h du\int_{\theta_1}^{\theta_2}\cos\frac{\theta}{2}d\theta\int_0^{\lambda}d\lambda=4r_0h\rho\int_0^{\lambda}\left(\sin\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\right)d\lambda. \tag{15}$$

8. To complete the evaluation of (15) it will be convenient to change variables. Consider the spherical triangles formed by the attracted and attracting points, the center of the attracting mass, and the points in which the arc θ cuts the circle bounding the mass. Thus in Figs. 1 and 2, let P be the attracting and A the attracted points, C the center of the attracting mass, and BDG the bounding circle. Then

$$\theta=AP \text{ and } \lambda=BAC.$$

Draw CE perpendicular to AB and put

$$AC=\alpha, \qquad BC=\beta,$$
$$PE=s, \qquad BE=s_0,$$
$$CE=p, \qquad AE=q,$$

From either figure

$$\theta=q+s, \qquad \theta_1=q-s_0, \qquad \theta_2=q+s_0,$$

whence

$$\sin\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}=2\cos\frac{q}{2}\sin\frac{s_0}{2}. \tag{16}$$

The right-angled spherical triangles of either figure give

$$\cos q=\frac{\cos\alpha}{\cos p}, \qquad \cos s_0=\frac{\cos\beta}{\cos p}, \qquad \sin p=\sin\alpha\sin\lambda. \tag{17}$$

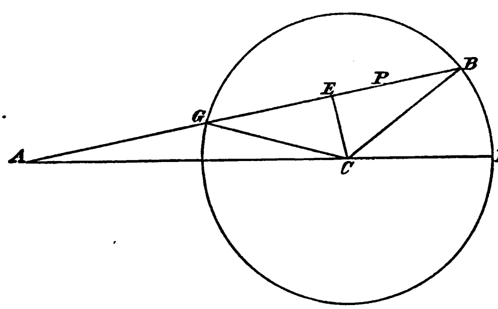

FIG. 1.

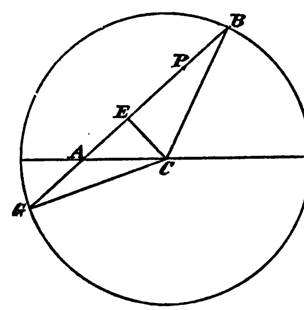

FIG. 2.

The first two of (17) give

$$2\cos^2\frac{q}{2}=1+\frac{\cos\alpha}{\cos p}, \qquad 2\sin^2\frac{s_0}{2}=1-\frac{\cos\beta}{\cos p},$$

whence

$$2\cos\frac{q}{2}\sin\frac{s_0}{2}=\frac{[(\cos p+\cos\alpha)(\cos p-\cos\beta)]^{\frac{1}{2}}}{\cos p}. \tag{18}$$

From the last of (17)

$$d\lambda=\frac{\cos p\,dp}{(\cos^2 p-\cos^2\alpha)^{\frac{1}{2}}}. \tag{19}$$

Now, the last of equations (17) and the diagrams show that the limits of p, corresponding to the limits of λ, are 0 and α or 0 and β, according as the attracted point is within or without the circle bounding the at-

tracting mass. Hence, if we denote the potentials in the two cases by V_1 and V_2, respectively, the equivalents in (15), (16), (18), and (19) give

$$V_1=4r_0h\rho\int_0^{\alpha}\left(\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp, \tag{20}$$

$$\alpha\leqq\beta;$$

$$V_2=4r_0h\rho\int_0^{\beta}\left(\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp, \tag{21}$$

$$\alpha\geqq\beta.$$

9. The integrals in these equations are in general elliptics of the third species. They may be evaluated by the usual processes applicable to elliptics, by series, or by mechanical quadrature.

The integral in (20) presents some apparent difficulty, since the element function is infinite at the upper limit, except when $\alpha=\beta$. Again, in case $\alpha=0$, this integral assumes the anomalous form

$$\int_0^0\left(\frac{1-\cos\beta}{1-1}\right)^{\frac{1}{2}}dp,$$

the value of which is $\pi\sin\frac{\beta}{2}$, as may be easily verified by means of (15), (16), and (18). These peculiar features may be removed by the following change of variables, which secures the same constant limits for both (20) and (21).

For brevity put

$$I_1=\int_0^{\alpha}\left(\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp, \tag{22}$$

$$I_2=\int_0^{\beta}\left(\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp. \tag{23}$$

Then, observing that

$$\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}=\frac{\sin^2\frac{\beta}{2}-\sin^2\frac{p}{2}}{\sin^2\frac{\alpha}{2}-\sin^2\frac{p}{2}},$$

put in I_1

$$\sin\frac{p}{2}=\sin\frac{\alpha}{2}\sin\gamma_1,$$

and in I_2

$$\sin\frac{p}{2}=\sin\frac{\beta}{2}\sin\gamma_2.$$

These give

$$dp=-\frac{2\sin\frac{\alpha}{2}\cos\gamma_1 d\gamma_1}{(1-\sin^2\frac{\alpha}{2}\sin^2\gamma_1)^{\frac{1}{2}}},$$

and

$$dp=\frac{2\sin\frac{\beta}{2}\cos\gamma_2 d\gamma_2}{(1-\sin^2\frac{\beta}{2}\sin^2\gamma_2)^{\frac{1}{2}}},$$

and the limits for both γ_1 and γ_2 are 0 and $\frac{\pi}{2}$. Therefore

$$I_1=\int_0^{\frac{\pi}{2}}\frac{2\sin\frac{\beta}{2}\left(1-\frac{\sin^2\frac{\alpha}{2}}{\sin^2\frac{\beta}{2}}\sin^2\gamma_1\right)^{\frac{1}{2}}d\gamma_1}{(1-\sin^2\frac{\alpha}{2}\sin^2\gamma_1)^{\frac{1}{2}}},\qquad (24)$$

$$\alpha\leqq\beta;$$

$$I_2=\int_0^{\frac{\pi}{2}}\frac{2\sin^2\frac{\beta}{2}\cos^2\gamma_2 d\gamma_2}{\sin\frac{\alpha}{2}\left(1-\frac{\sin^2\frac{\beta}{2}}{\sin^2\frac{\alpha}{2}}\sin^2\gamma_2\right)^{\frac{1}{2}}\left(1-\sin^2\frac{\beta}{2}\sin^2\gamma_2\right)^{\frac{1}{2}}},\qquad (25)$$

$$\alpha\geqq\beta.$$

10. Some special values of the integrals (22) to (25) and the corresponding potentials (20) and (21) are worth deriving. These values are:

(*a*) For a point of the disturbed surface at the center of the disturbing mass, $\alpha=0$, and (24) gives

$$I_1=2\sin\frac{\beta}{2}\int_0^{\frac{\pi}{2}}d\gamma_1=\pi\sin\frac{\beta}{2},\qquad (26)$$

and the corresponding value of the potential is

$$V_1=4r_0h\rho\pi\sin\frac{\beta}{2}.\qquad (27)$$

(*b*) For a point at the border of the disturbing mass, $\alpha=\beta$, and hence from (24) and (25)

$$I_1=I_2=2\text{ arc sin}\left[\sin\frac{\beta}{2}\sin\gamma\right]_{\gamma=0}^{\gamma=\frac{\pi}{2}}=\beta,\qquad (28)$$

a result which is reached more readily from (22) or (23). The corresponding value of the potential is

$$V_1 = V_2 = 4r_0 h\rho\beta. \tag{29}$$

(*c*) For a point of the disturbed surface 180° from the center of the disturbing mass, $\alpha=\pi$, and (25) gives

$$I_2 = 2\int_0^{\frac{\pi}{2}} \frac{\sin^2\frac{\beta}{2}\cos^2\gamma_2 d\gamma_2}{1-\sin^2\frac{\beta}{2}\sin^2\gamma_2} = 2\int_0^{\frac{\pi}{2}} \frac{\left(1-\cos^2\frac{\beta}{2}\right)d\gamma_2}{1+\cos^2\frac{\beta}{2}\tan^2\gamma_2}$$

$$= \pi - 2\cos\frac{\beta}{2}\,\text{arc tan}\Big[\tan\Big(\cos\frac{\beta}{2}\tan\gamma_2\Big)\Big]_0^{\frac{\pi}{2}}$$

$$= \pi\left(1-\cos\frac{\beta}{2}\right) = 2\pi\sin^2\frac{\beta}{4}, \tag{30}$$

and the potential is

$$V_2 = 8r_0 h\rho\pi\sin^2\frac{\beta}{4}. \tag{31}$$

(*d*) Suppose $\beta=\pi$; in other words, let the attracting mass cover the whole sphere. Then (24) gives

$$I_1 = \pi, \tag{32}$$

and we have the following well-known approximate value for the potential of a spherical shell for a point on its interior or exterior surface[1], viz:

$$4r_0 h\rho\pi. \tag{33}$$

This result follows also from (27) or (29) if we make $\beta=\pi$.

IV. DEGREE OF APPROXIMATION OF THE EXPRESSIONS FOR THE POTENTIAL V.

11. In deriving the expressions (20) and (21) for the potential of the disturbing mass, it was assumed that a sufficient degree of approximation is attained if quantities of the orders $\frac{u}{r_0}$, $\frac{v}{r_0}$, and upwards are neglected. The grounds of this assumption need to be examined with some care. For this purpose we shall derive the exact expressions, in form at least, for the potential of the disturbing mass at its center, at its border, and at 180° from its center. A comparison of these exact values with the approximate values given by (27), (29), and (31) will show the order of approximation of (20) and (21).

[1] See equation 34.

We will first write down the expression for the potential of a complete spherical shell, of uniform thickness and density, for a point within its bounding surfaces. This expression will be useful as a check on formulas relating to partial shells.

Let the radius of the interior surface of the complete shell be r_0, the uniform thickness h, the uniform density ρ, and the distance of the attracted point from the center of the shell r_0+v.

Then the potential is[1]

$$V=4r_0h\rho\pi\left(1+\frac{h}{2(r_0+v)}-\frac{v^2}{2(r_0+v)h}+\frac{hv}{2r_0(r_0+v)}-\frac{v^3}{6r_0(r_0+v)h}\right). \quad (34)$$

12. Let the notation be the same as that used heretofore; i. e., let r be the radius-vector of any point of the attracting mass, r' the radius-vector of any point of the disturbed surface, ρ the density, h the thickness, and β the angular extent of the mass; and θ the angular distance between the attracted and attracting points. For points of the disturbed surface lying above the undisturbed surface, r will be less than $r'=r_0+v$ over the range r_0 to r', and r will be greater than r' over the range r' to r_0+h. Bearing these facts in mind it follows that the exact value of the potential of the disturbing mass for the point where its axis pierces the disturbed surface is

$$V=2\rho\pi\int_{r_0}^{r_0+v} r^2dr\int_0^{\beta}\frac{\sin\theta d\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}+$$

$$2\rho\pi\int_{r_0+v}^{r_0+h} r^2dr\int_0^{\beta}\frac{\sin\theta d\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}$$

$$=2\rho\pi\int_{r_0}^{r_0+v}\left(\sqrt{4rr'\sin^2\frac{\beta}{2}+(r'-r)^2}-(r'-r)\right)\frac{rdr}{r'}+$$

$$2\rho\pi\int_{r_0+v}^{r_0+h}\left(\sqrt{4rr'\sin^2\frac{\beta}{2}+(r-r')^2}-(r-r')\right)\frac{rdr}{r'}.$$

---

[1] $2\pi\rho\left((r_0+h)^2-\frac{1}{3}(r_0+v)^2-\frac{2}{3}\frac{r_0^3}{r_0+v}\right)$

See Price's Calculus, vol. 3, p. 299.

Since

$$\sqrt{4rr'\sin^2\frac{\beta}{2}+(r-r')^2}=2\sin\frac{\beta}{2}\sqrt{rr'}\left(1+\frac{(r-r')^2}{8rr'\sin^2\frac{\beta}{2}}-\ldots\right)=$$

$$2r\sin\frac{\beta}{2}\left(1+\frac{(r-r')^2}{8r'r\sin^2\frac{\beta}{2}}-\ldots\right)\left(1-\frac{r-r'}{2r}+\ldots\right),$$

we find by expansion, integration, and reduction, to terms of the first order inclusive,

$$V=4r_0h\rho\pi\sin\frac{\beta}{2}\left\{\begin{array}{l}1+\dfrac{3h\sin\frac{\beta}{2}-h}{4(r_0+v)\sin\frac{\beta}{2}} \\ +\dfrac{v\left(1-\sin\frac{\beta}{2}\right)}{2(r_0+v)\sin\frac{\beta}{2}}-\dfrac{v^2}{2(r_0+v)h\sin\frac{\beta}{2}}\end{array}\right\}. \qquad (35)$$

If we make $\beta=\pi$ this expression agrees with (34) to terms of the second order.

Equation (35), it will be observed, differs from (27) by certain terms which must be small unless h and v are very large. In one of the most important applications discussed in the sequel, h=10,000 feet, v=3,000 feet, and $\beta=38°$. With these values, since r_0 is in round numbers 21,000,000 feet, the quantity within the parentheses of (35) differs from unity by less than $\frac{1}{10000}$. If $\beta=60°$, which is (see section 23) the angular extent of mass required to produce the maximum elevation of the disturbed surface at the center of the mass, the quantity within the brackets of (35) exceeds unity by less than $\frac{1}{8000}$, using the above values of h and v.

13. Similarly, if the attracted point be at the border of the attracting mass, the exact value of the potential is

$$V=2\rho\int_{r_0}^{r_0+v}\frac{r}{r'}dr\int_0^{\frac{\pi}{2}}\sqrt{(r'-r)^2+4rr'\sin^2\frac{\theta_2}{2}}\,.\,d\lambda$$

$$+2\rho\int_{r_0+v}^{r_0+h}\frac{r}{r'}dr\int_0^{\frac{\pi}{2}}\sqrt{(r-r')^2+4rr'\sin^2\frac{\theta_2}{2}}\,.\,d\lambda$$

$$-\rho\pi\int_{r_0}^{r_0+v}(r'-r)\frac{r}{r'}dr-\rho\pi\int_{r_0+v}^{r_0+h}(r-r')\frac{r}{r'}dr,$$

in which

$$\sin^2\frac{\theta_2}{2}=\frac{\sin^2\beta\cos^2\lambda}{1-\sin^2\beta\sin^2\lambda}.$$

Introducing this value, the first of the above integrals relative to λ becomes

$$(r'+r)\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin^2\beta\cos^2\lambda+\left(\frac{r'-r}{r'+r}\right)^2\cos^2\beta}{1-\sin^2\beta\sin^2\lambda}}d\lambda.$$

Since the numerator of the element function in this integral is greater than $\sin\beta\cos\lambda$ and less than $\sin\beta\cos\lambda+\frac{r'-r}{r'+r}\cos\beta$, the value of the integral lies between $(r'+r)\beta$ and

$$(r'+r)\beta\left(1+\frac{r'-r}{r'+r}\,\frac{\cos\beta}{\beta}\int_0^{\frac{\pi}{2}}\frac{d\lambda}{\sqrt{1-\sin^2\beta\sin^2\lambda}}\right).$$

Suppose the exact value of this integral is

$$(r'+r)\beta\left\{1+Q_1\frac{r'-r}{r'+r}\right\}.$$

Likewise, represent the exact value of the second integral relative to λ by

$$(r+r')\beta\left\{1+Q_2\frac{r-r'}{r+r'}\right\}.$$

Then to terms of the first order inclusive the above expression for the potential becomes

$$V=4r_0h\rho\beta\left\{1+\frac{h(3-Q_2)}{4(r_0+v)}-\frac{v(1+Q_2)}{2(r_0+v)}+\frac{v^2(Q_1+Q_2)}{4r_0h}-\frac{\pi}{\beta}\cdot\frac{h^2-2hv+2v^2}{8(r_0+v)h}\right\}. \quad (36)$$

The first term of this agrees with (29).

The quantities Q_1 and Q_2 lie between

$$0 \text{ and } \frac{\cos\beta}{\beta}\int_0^{\frac{\pi}{2}}\frac{d\lambda}{\sqrt{1-\sin^2\beta\sin^2\lambda}}.$$

When $\beta=\frac{\pi}{2}$, or when the attracting stratum covers a whole hemisphere, $Q_1=Q_2=0$, and (36) becomes

$$V=2r_0h\rho\pi\left\{1+\frac{h}{2(r_0+v)}-\frac{v^2}{2(r_0+v)h}\right\}. \quad (36')$$

This agrees, as it should, to terms of the second order with the half of (34). For $h=10{,}000$ feet and $v=3{,}000$ feet the quantity in the parenthesis exceeds unity by less than $\frac{1}{4000}$. When $\beta<\frac{\pi}{2}$, Q_1 and Q_2 will exceed 0, and the degree of approximation of (36) will be somewhat higher than that of (36′).

14. For a point of the disturbed surface 180° from the center of the disturbing mass the exact value of the potential is

$$V=2\rho\pi\int_{r_0}^{r_0+h} r^2dr\int_0^{\beta}\frac{\sin\theta d\theta}{\sqrt{r^2+r'^2+2rr'\cos\theta}}$$

$$=2\rho\pi\int_{r_0}^{r_0+h}(r+r'-\sqrt{r^2+r'^2+2rr'\cos\beta})\frac{r}{r'}dr.$$

When

$$(r-r')^2<4rr'\cos^2\frac{\beta}{2},$$

which is the only case we need consider,

$$\sqrt{r^2+r'^2+2rr'\cos\beta}=2\cos\frac{\beta}{2}\sqrt{rr'}\left\{1+\frac{(r-r')^2}{8rr'\cos^2\frac{\beta}{2}}-\dots\right\}$$

$$=2r\cos\frac{\beta}{2}\left\{1+\frac{(r-r')^2}{8rr'\cos\frac{\beta}{2}}-\dots\right\}\left\{1-\frac{r-r'}{2r}+\dots\right\}.$$

Hence, expanding, integrating, and reducing, there results to terms of the first order inclusive,

$$V=8r_0h\rho\pi\sin^2\frac{\beta}{2}\left\{1+\frac{3h-2v}{4(r_0+v)}\right\}, \tag{37}$$

the first term of which agrees with (31). Using the values $r_0=21{,}000{,}000$ feet, $h=10{,}000$ feet, and $v=-1{,}000$ feet (v being here intrinsically negative), the factor in the brackets of (37) exceeds unity by $\frac{1}{2600}$.

V. DEVELOPMENT OF POTENTIAL V IN SERIES OF SPHERICAL HARMONICS.

15. The preceding expressions for the potential of the attracting mass, namely, V, as defined by equations (20) and (21), are sufficient for most of the applications to be considered in the sequel. They possess the obvious advantage of a compact integral form. For some purposes, however, it will be desirable to have V expressed in a series of spherical harmonics or Laplace's functions. We may thereby arrive at equa-

tions (20) and (21) by a process differing from that followed in Article III and establish a useful harmonic development of the elliptic integrals I_1 and I_2 of equations (22) to (25).

16. Expressions fulfilling the present requirements may be derived from equation (10) by expanding D^{-1} in a series of ascending powers of $\frac{r}{r'}$ and $\frac{r'}{r}$. Thus from equation (9) we have

$$\frac{1}{D}=\frac{1}{r'}\left[P_0+P_1\left(\frac{r}{r'}\right)+P_2\left(\frac{r}{r'}\right)^2+\ldots\right] \text{ when } r<r',$$

$$\frac{1}{D}=\frac{1}{r}\left[P_0+P_1\left(\frac{r'}{r}\right)+P_2\left(\frac{r'}{r}\right)^2+\ldots\right] \text{ when } r>r'.$$

In these equations P_0, P_1, P_2, etc., are Laplace's coefficients of the zero, first, second, etc., order, respectively. They are functions of the angular coördinates θ, θ', λ, and λ' only.

Taking, now, the center of the sphere of reference as the origin of coördinates, and supposing the line from which θ and θ' are reckoned to pass through the center of the attracting mass, we shall have for a mass of uniform thickness h, all integrations in (10) independent. For that part of the disturbed surface which lies above the undisturbed surface $r'=r_0+v$ will fall between the extreme values of r, which are r_0 and r_0+h; and hence $r<r'$ for values between r_0 and r_0+v, $r>r'$ for values between r_0+v and r_0+h. For that part of the disturbed surface lying below the undisturbed surface $r>r'$. In both cases the limits of θ are 0, and the angular radius β of the attracting mass and the limits of λ are 0 and 2π. Therefore for that part of the disturbed surface lying above the undisturbed surface equation (10) gives

$$\begin{aligned} V=\rho\int_0^{\beta}\sin\theta d\theta\int_0^{2\pi}d\lambda\int_{r_0}^{r_0+v}\left[P_0+P_1\left(\frac{r}{r'}\right)+P_2\left(\frac{r}{r'}\right)^2+\ldots\right]\frac{r^2dr}{r'} \\ +\rho\int_0^{\beta}\sin\theta d\theta\int_0^{2\pi}d\lambda\int_{r_0+v}^{r_0+h}\left[P_0+P_1\left(\frac{r'}{r}\right)+P_2\left(\frac{r'}{r}\right)^2+\ldots\right]rdr; \end{aligned} \quad (38)$$

$$r'=r_0+v.$$

Likewise for that part of the disturbed surface which lies below the undisturbed surface, equation (10) gives

$$V=\rho\int_0^{\beta}\sin\theta d\theta\int_0^{2\pi}d\lambda\int_{r_0}^{r_0+h}\left[P_0+P_1\left(\frac{r'}{r}\right)+P_2\left(\frac{r'}{r}\right)^2+\ldots\right]rdr; \quad (39)$$

$$r'=r_0+v.$$

For brevity, let the integrals with respect to r in (38) be denoted as follows:

$$
\left.\begin{aligned}
J_0&=\frac{(r_0+v)^3-r_0^3}{3(r_0+v)},\\
J_1&=\frac{(r_0+v)^4-r_0^4}{4(r_0+v)^2},\\
J_2&=\frac{(r_0+v)^5-r_0^5}{5(v_0+v)^3},\\
&\cdots\cdots\\
J_i&=\frac{(r_0+v)^{i+3}-r_0^{i+3}}{(i+3)(r_0+v)^{i+1}}.
\end{aligned}\right\}\qquad(40)
$$

$$
\left.\begin{aligned}
J_0'&=\frac{(r_0+h)^2-(r_0+v)^2}{2},\\
J_1'&=(r_0+h)\,(h-v),\\
J_2'&=(r_0+v)^2\log_e\frac{r_0+h}{r_0+v},\\
&\cdots\cdots\\
J_i'&=\frac{(r_0+v)^i}{i-2}\Big[(r_0+v)^{-(i-2)}-(r_0+h)^{-(i-2)}\Big].
\end{aligned}\right\}\qquad(41)
$$

The ambiguous form which J_i' assumes when $i=2$ receives its proper interpretation in the third of (41).

Similarly, let the integrals with respect to r in (39) be denoted thus:

$$
\left.\begin{aligned}
J_0''&=\frac{(r_0+h)^2-r_0^2}{2},\\
J_1''&=(r_0+v)h,\\
J_2''&=(r_0+v)^2\log_e\frac{r_0+h}{r_0},\\
&\cdots\cdots\\
J_i''&=\frac{(r_0+v)^i}{i-2}\Big[r_0^{-(i-2)}-(r_0+h)^{-(i-2)}\Big].
\end{aligned}\right\}\qquad(42)
$$

Substituting these equivalents of (40), (41), and (42) in (38) and (39) the latter become, respectively,

$$V=\rho\int_0^\beta \sin\theta d\theta\int_0^{2\pi}[(J_0+J_0')P_0+(J_1+J_1')P_1+\ .\ \ .\ \ .]d\lambda,\qquad(43)$$

$$V=\rho\int_0^\beta \sin\theta d\theta\int_0^{2\pi}[J_0''P_0+J_1''P_1+\ .\ \ .\ \ .]d\lambda.\qquad(44)$$

Now, it is known from the theory of spherical harmonics[1] that the general value of P_i is

$P_i = f_i(\cos\theta) f_i(\cos\theta') +$ terms multiplied by cosines of multiples of $(\lambda - \lambda')$.

Since

$$\int_0^{2\pi} \cos i(\lambda-\lambda')d\lambda = 0,$$

we have to deal only with the first term of P_i. The function of $\cos\theta$ or $\cos\theta'$ involved in this first term is defined as follows:[2]

$$\mu = \cos\theta,$$

$$f_i(\cos\theta) = f_i(\mu) = \frac{1}{2^i \,.\, 1 \,.\, 2 \,.\,.\, i} \cdot \frac{d^i(\mu^2-1)^i}{d\mu^i}.$$

The following important relation exists between any three consecutive values of $f_i(\mu)$, viz:[3]

$$f_i(\mu) = \frac{2i-1}{i}\mu f_{i-1}(\mu) - \frac{i-1}{i} f_{i-2}(\mu). \tag{45}$$

The remaining integrals in (43) and (44) are therefore of the form

$$f_i(\cos\theta')\int_0^\beta f_i(\cos\theta)\sin\theta d\theta \int_0^{2\pi} d\lambda = 2\pi f_i(\mu')\int_{\cos\beta}^1 f_i(\mu)d\mu.$$

Let

$$F_i(\beta) = \int_{\cos\beta}^1 f_i(\mu)d\mu. \tag{46}$$

The known value of this integral is[4]

$$F_i(\beta) = \frac{1}{2i+1}\Big[f_{i-1}(\cos\beta) - f_{i+1}(\cos\beta)\Big]. \tag{47}$$

The values of V in (43) and (44) may now be written thus, replacing θ' by α, θ' or α being the angular distance of the attracted point from the center of the attracting mass:

$$V = 2\rho\pi \sum_{i=0}^{i=\infty} [(J_i + J_i')f_i(\cos\alpha)F_i(\beta)] \tag{48}$$

for points of disturbed surface above undisturbed.

$$V = 2\rho\pi \sum_{i=0}^{i=\infty} [J_i'' f_i(\cos\alpha)F_i(\beta)] \tag{49}$$

for points of disturbed surface below undisturbed.

[1] See Heine, Handbuch der Kugelfunctionen, Theorie und Anwendungen, Erster Band. Second edition, G. Reimer, Berlin, 1878.

[2] Heine, p. 19. [3] Heine, p. 91. [4] Heine, p. 93.

17. The values of V in the last two equations are exact, but their applicability to the problems we have to consider is limited by the slow convergence of the series in the second members. In these problems r_0, r_0+v, and r_0+h are nearly equal, and hence the convergence of the series depends almost wholly on the convergence of the functions of α and β.

If we neglect terms of the order h/r_0, v/r_0, and upwards, equations (48) and (49) become identical and equivalent to (20) and (21). To show these facts we expand J_i and J_i' of (40) and (41) and obtain

$$J_i+J_i'=r_0h\left[1-\frac{1}{2r_0}\left\{(i-1)h-2iv+\left(i+\frac{1}{2}\right)\frac{v^2}{h}\right\}+O_2\right], \tag{50}$$

in which O_2 represents terms of the second order. Likewise, (42) gives

$$J_i''=r_0h\left[1-\frac{1}{2}(i-1)\frac{h}{r_0}+i\frac{v}{r_0}+O_2\right]. \tag{51}$$

Hence, to terms of the first order we have[1]

$$J_i+J_i'=J_i''=r_0h,$$

and (48) and (49) become

$$V=2r_0h\rho\pi\sum_{i=0}^{i=\infty}f_i(\cos\alpha)F_i(\beta). \tag{52}$$

Now, if $\cos\psi=\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos(\lambda-\lambda')$,

$$2\pi\sum_{i=0}^{i=\infty}f_i(\cos\alpha)F_i(\beta)=\int_0^\beta\int_0^{2\pi}\frac{\sin\theta d\theta d\lambda}{\sqrt{1-2\cos\psi+1}}.$$

But this integral, as shown by the transformation in section **3**, is equivalent to

$$4\int_0^\alpha\left(\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp \text{ or } 4\int_0^\beta\left(\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp,$$

according as α is less or greater than β.

---

[1] This inference from (50) and (51) does not appear to be quite satisfactory. For large values of i, J_i+J_i' and J_i'' are less than r_0h; they are each 0 for $i=\infty$. The influence of the too great factor r_0h in the higher terms of (52) is, however, counteracted by the small factors $f_i(\cos\alpha)F_i(\beta)$ in those terms; and that the order of approximation secured in (52) is sufficient is evident from the equivalence of (52) with (20) and (21), whose order of approximation has been investigated in Article IV.

This establishes the equivalence of (52) with (20) and (21), and furnishes the following development in polar harmonics of the elliptic integrals in (22) to (25):

$$
\left.
\begin{array}{l}
I_1=\displaystyle\int_0^{\alpha}\left(\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp=\frac{\pi}{2}\sum_{i=0}^{i=\infty}f_i(\cos\alpha)F_i(\beta),\\
\qquad\qquad\qquad\qquad\alpha\leqq\beta;\\
I_2=\displaystyle\int_0^{\beta}\left(\frac{\cos p-\cos\beta}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp=\frac{\pi}{2}\sum_{i=0}^{i=\infty}f_i(\cos\alpha)F_i(\beta),\\
\qquad\qquad\qquad\qquad\alpha\geqq\beta.
\end{array}
\right\}\quad(53)
$$

In order to show the form of the second numbers of equations (52) and (53) we give below the first four values of $f_i(\cos\alpha)$ and $F_i(\beta)$, respectively. The series of values may be easily extended by means of the relations in (45) and (47).

$$
\begin{array}{ll}
f_0(\cos\alpha)=1, & F_0(\beta)=1-\cos\beta=2\sin^2\dfrac{\beta}{2},\\
f_1(\cos\alpha)=\cos\alpha, & F_1(\beta)=\frac{1}{2}(1-\cos^2\beta)=\frac{1}{2}\sin^2\beta,\\
f_2(\cos\alpha)=\frac{1}{2}(3\cos^2\alpha-1), & F_2(\beta)=\frac{1}{2}(\cos\beta-\cos^3\beta)=\frac{1}{2}\sin^2\beta\cos\beta,\\
f_3(\cos\alpha)=\frac{1}{2}(5\cos^3\alpha-3\cos\alpha); & F_3(\beta)=\frac{1}{8}(6\cos^2\beta-5\cos^4\beta-1).
\end{array}
$$

By means of these values equation (52) may be written thus:

$$
V=2r_0h\rho\pi\left\{
\begin{array}{l}
+2\sin^2\dfrac{\beta}{2}+\dfrac{1}{2}\cos\alpha\sin^2\beta\\
+\frac{1}{4}(3\cos^2\alpha-1)\sin^2\beta\cos\beta\\
+\frac{1}{16}(5\cos^3\alpha-3\cos\alpha)(6\cos^2\beta-5\cos^4\beta-1)\\
+\ \cdot\ \cdot\ \cdot
\end{array}
\right\}.\quad(54)
$$

VI. EFFECT OF REARRANGED FREE WATER.

18. In case the disturbing mass is as large as the supposed ice mass of the glacial epoch, the attraction of the rearranged free water on itself may be appreciable. To determine the exact effect of this attraction would be a work of great difficulty even if we had the requisite information, namely, an accurate knowledge of the complicated shapes of the continents and sea bottom. But we may determine an effect which will exceed the probable actual effect by supposing the whole surface of the earth covered with a film of water free to assume the

proper form for equilibrium under the given forces. To fit this ideal case formula (6) may be modified in the following manner:

Let the potential V in (6) be replaced by $V+\Delta V$, where ΔV is the potential due to the rearrangement of the water. Likewise replace the constant V_0 by $V_0+\Delta V_0$. Then, if Δv denote the corresponding change in v, equation (6) gives

$$v+\Delta v=\frac{3}{4r_0\rho_m\pi}(V+\Delta V-V_0-\Delta V_0). \tag{55}$$

Now, $v+\Delta v$ may be expressed by a series of Laplace's functions (which are in this case polar harmonics) thus:

$$v+\Delta v=r_0c_0(Z_0+Z_1+Z_2+\ .\ .\ .\ ),$$

in which c_0 is a constant of small numerical value; and hence, denoting the density of sea water by ρ_w, we have, as shown by Laplace,[1]

$$\Delta V=4r_0^2c_0\rho_w\pi(Z_0+\frac{1}{3}Z_1+\frac{1}{5}Z_2\ .\ .\ .\ \frac{1}{2i+1}Z_i+\ .\ .\ .\ ).$$

V has already been expressed, equation (54), in a series of Laplace's functions. Denoting these functions in (54) for brevity by Y_0, Y_1, Y_2, etc.,

$$V=2r_0h\rho\pi(Y_0+Y_1+Y_2+\ .\ .\ .\ ).$$

Substituting these values of V, ΔV, and $v+\Delta v$ in (55) there results, if we make the obviously permissible substitution

$$\frac{3}{2}h\frac{\rho}{\rho_m}U_0=\frac{3}{4r_0\rho_m\pi}(V_0+\Delta V_0),$$

the following equation:

$$\left.\begin{array}{l}
r_0c_0Z_0\left(1-\dfrac{3\rho_w}{\rho_m}\right)-\dfrac{3}{2}h\dfrac{\rho}{\rho_m}(Y_0-U_0) \\
+r_0c_0Z_1\left(1-\dfrac{3\rho_w}{3\rho_m}\right)-\dfrac{3}{2}h\dfrac{\rho}{\rho_m}Y_1 \\
+r_0c_0Z_2\left(1-\dfrac{3\rho_w}{5\rho_m}\right)-\dfrac{3}{2}h\dfrac{\rho}{\rho_m}Y_2 \\
.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \\
+r_0c_0Z_i\left(1-\dfrac{3\rho_w}{(2i+1)\rho_m}\right)-\dfrac{3}{2}h\dfrac{\rho}{\rho_m}Y_i \\
+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .
\end{array}\right\}=0.$$

According to the theory of Laplace's functions we must have in this equation the sums of the functions of the same order separately equal

[1] Mécanique Céleste, Book III, Chap. II.

to zero, which amounts to placing each line in the equation equal to zero. Thus we find

$$r_0c_0Z_0=\frac{\frac{3}{2}h\frac{\rho}{\rho_m}}{1-\frac{3\rho_w}{\rho_m}}(Y_0-U_0),$$

$$r_0c_0Z_1=\frac{\frac{3}{2}h\frac{\rho}{\rho_m}}{1-\frac{3\rho_w}{3\rho_m}}Y_1$$

. . . .

$$r_0c_0Z_i=\frac{\frac{3}{2}h\frac{\rho}{\rho_m}}{1-\frac{3\rho_w}{(2i+1)\rho_m}}Y_i;$$

whence by summation

$$v+\varDelta v=r_0c_0\sum_{i=0}^{i=\infty}Z_i=\frac{3}{2}h\frac{\rho}{\rho_m}\left(\sum_{i=0}^{i=\infty}\frac{Y_i}{1-\frac{3}{2i+1}\frac{\rho_w}{\rho_m}}-\frac{U_0}{1-3\frac{\rho_w}{\rho_m}}\right). \tag{56}$$

This equation expresses the total effect of the disturbing mass in altering the sea level, v being the effect which would result if the ocean were an infinitely rare fluid, and $\varDelta v$ being the increase over v which would result under the assumed conditions. Obviously, v and $\varDelta v$ may be expressed separately. Thus

$$\varDelta v=\frac{3}{2}h\frac{\rho}{\rho_m}\left(\sum_{i=0}^{i=\infty}\frac{3Y_i}{(2i+1)\frac{\rho_m}{\rho_w}-3}-\frac{3U_0}{\frac{\rho_m}{\rho_w}-3}\right). \tag{57}$$

VII. EVALUATION OF CONSTANTS V_0 AND U_0.

19. We proceed now to determine the constants V_0 of equation (6) and U_0 of equations (56) and (57).

It has already been stated that these constants are to be determined from the condition of equality in volumes contained by the disturbed and undisturbed surfaces, a condition whose analytical statement is, to terms of the order we neglect,

$$2r_0^2\pi\int_0^\pi v\sin\alpha d\alpha=0.$$

Substituting the value of v from equation (6) in this, there results

$$\int_0^\pi V \sin \alpha d\alpha - V_0 \int_0^\pi \sin \alpha d\alpha = 0.$$

whence

$$V_0 = \frac{1}{2} \int_0^\pi V \sin \alpha d\alpha. \tag{58}$$

(*a*) The easiest way to evaluate this integral is to substitute for V its value given by equation (54). We get, then, at once

$$V_0 = 4 r_0 h \rho \pi \sin^2 \frac{\beta}{2}, \tag{59}$$

since by the theory of spherical harmonics all terms of the series except the first vanish in the integration. For the same reason, if we apply the condition

$$\int_0^\pi (v + \Delta v) \sin \alpha d\alpha = 0$$

to equation (56), it will appear that

$$U_0 = Y_0 = 2 \sin^2 \frac{\beta}{2}. \tag{60}$$

(*b*) The value of V_0 may also be found by the following process, which is chiefly interesting on account of its complication as compared with the process used above. For points within the perimeter of the attracting mass replace V in (58) by V_1 of (20), and for points outside the perimeter replace V by V_2 of (21). Making these substitutions, there results

$$V_0 = 2 r_0 h \rho \left(\int_0^\beta I_1 \sin \alpha d\alpha + \int_\beta^\pi I_2 \sin \alpha d\alpha \right). \tag{61}$$

Substituting the value of I_1 from (24),

$$\int_0^\beta I_1 \sin \alpha d\alpha = 4 \int_0^{\frac{\pi}{2}} \frac{d\gamma_1}{\sin^2 \gamma_1} \int_0^\beta \left(\frac{\sin^2 \frac{\beta}{2} - \sin^2 \frac{\alpha}{2} \sin^2 \gamma_1}{1 - \sin^2 \frac{\alpha}{2} \sin^2 \gamma_1} \right)^{\frac{1}{2}} \sin^2 \gamma_1 d \left(\sin^2 \frac{\alpha}{2} \right).$$

Put

$$t^2 = \frac{\sin^2 \frac{\beta}{2} - \sin^2 \frac{\alpha}{2} \sin^2 \gamma_1}{1 - \sin^2 \frac{\alpha}{2} \sin^2 \gamma_1}.$$

Then the last integral becomes

$$-4\cos^2\frac{\beta}{2}\int_0^{\frac{\pi}{2}}\frac{d\gamma_1}{\sin^2\gamma_1}\int_{t_1}^{t_2}\frac{2t^2dt}{(t^2-1)^2}.$$

$$=4\cos^2\frac{\beta}{2}\int_0^{\frac{\pi}{2}}\left[\frac{t_2}{t_2^2-1}-\frac{t_1}{t_1^2-1}+\frac{1}{2}\log_e\frac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}\right]\frac{d\gamma_1}{\sin^2\gamma_1},$$

in which

$$t_1=\sin\frac{\beta}{2},$$

and

$$t_2=\frac{\sin\frac{\beta}{2}\cos\gamma_1}{\left(1-\sin^2\frac{\beta}{2}\sin^2\gamma_1\right)^{\frac{1}{2}}}.$$

Substituting these limits in the non-logarithmic part of the integral, it becomes

$$4\sin^2\frac{\beta}{2}\left\{\int_0^{\frac{\pi}{2}}\frac{d\gamma_1}{\sin\frac{\beta}{2}\sin^2\gamma_1}-\frac{1}{\sin\frac{\beta}{2}}\int_0^{\frac{\pi}{2}}\left(1-\sin^2\frac{\beta}{2}\sin^2\gamma_1\right)^{\frac{1}{2}}\frac{\cos\gamma_1}{\sin^2\gamma_1}d\gamma_1\right\}$$

$$+2\sin^2\frac{\beta}{2}\cot^2\frac{\beta}{2}\int_0^{\frac{\pi}{2}}\frac{d\gamma}{\sin^2\gamma_1}\log_e\frac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}.$$

Integrating by parts all terms of this expression except the first, we get

$$4\sin^2\frac{\beta}{2}\left\{\begin{array}{l}+\dfrac{(1-\sin^2\frac{\beta}{2}\sin^2\gamma_1)^{\frac{1}{2}}}{\sin\frac{\beta}{2}\sin\gamma_1}-\dfrac{\cot\gamma_1}{\sin\frac{\beta}{2}}\\ +\text{arc}\sin\left(\sin\frac{\beta}{2}\sin\gamma_1\right)\\ -\frac{1}{2}\cot^2\frac{\beta}{2}\left(\log_e\dfrac{(1+t_2)(1-t_1)}{(1-t_2)(1+t_1)}\right)\cot\gamma_1\\ -\cot^2\frac{\beta}{2}\,\text{arc}\sin\left(\sin\frac{\beta}{2}\sin\gamma_1\right)\end{array}\right\}_0^{\frac{\pi}{2}}.$$

This gives

$$\int_0^{\beta}I_1\sin\alpha d\alpha=2(\sin\beta-\beta\cos\beta). \tag{62}$$

The second integral in (61) becomes by substitution of the value of I_2 from (25)

$$\int_\beta^\pi I_2 \sin \alpha d\alpha$$

$$=4 \sin^2 \frac{\beta}{2} \int_0^{\frac{\pi}{2}} \cos^2 \gamma_2 d\gamma_2 \int_\beta^\pi \frac{d\left(\sin^2 \frac{\alpha}{2}\right)}{\left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2} \sin^2 \gamma_2\right)^{\frac{1}{2}} \left(1-\sin^2 \frac{\beta}{2} \sin^2 \gamma_2\right)^{\frac{1}{2}}}$$

$$=8 \sin^2 \frac{\beta}{2} \int_0^{\frac{\pi}{2}} \left\{ \cos^2 \gamma_2 - \frac{\sin \frac{\beta}{2} \cos \gamma_2}{\left(1-\sin^2 \frac{\beta}{2} \sin^2 \gamma_2\right)^{\frac{1}{2}}} + \frac{\sin \frac{\beta}{2} \sin^2 \gamma_2 \cos \gamma_2}{\left(1-\sin^2 \frac{\beta}{2} \sin^2 \gamma_2\right)^{\frac{1}{2}}} \right\} d\gamma_2$$

$$=4 \sin^2 \frac{\beta}{2} \left(\begin{array}{l} \gamma_2 + \frac{1}{2} \sin 2\gamma_2 - 2 \text{ arc sin} \left(\sin \frac{\beta}{2} \sin \gamma_2\right) \\ -\frac{\left(1-\sin^2 \frac{\beta}{2} \sin^2 \gamma_2\right)^{\frac{1}{2}} \sin \gamma_2}{\sin \frac{\beta}{2}} \\ +\frac{\text{arc sin} \left(\sin \frac{\beta}{2} \sin \gamma_2\right)}{\sin^2 \frac{\beta}{2}} \end{array} \right)_0^{\frac{\pi}{2}}.$$

This gives

$$\int_\beta^\pi I_2 \sin \alpha d\alpha = 2\left(\pi \sin^2 \frac{\beta}{2} - \sin \beta + \beta \cos \beta\right). \tag{63}$$

The sum of (62) and (63) is

$$2\pi \sin^2 \frac{\beta}{2},$$

which, substituted in (61), gives for V_0 the same value as (59).

VIII. EQUATIONS OF DISTURBED SURFACE.

20. By reference now to equations (3), (6), (20) to (23), and (59) we find for the equation of the disturbed surface when the effect of the rearranged water is neglected

$$v = 3h \frac{\rho}{\rho_m} \left(\frac{I}{\pi} - \sin^2 \frac{\beta}{2}\right). \tag{64}$$

The corresponding expression in polar harmonics is [see equations (56), (57), and (60)]

$$v=\frac{3}{2}h\frac{\rho}{\rho_m}\sum_{i=1}^{i=\infty}[Y_i=f_i(\cos\alpha)\,F_i(\beta)]. \tag{65}$$

Under the assumption that the water covers the whole sphere and is free to adjust itself as stated in Article VI, the equation to the disturbed surface is

$$\begin{aligned} v+\Delta v&=\frac{3}{2}h\frac{\rho}{\rho_m}\sum_{i=1}^{i=\infty}\left[\frac{f_i(\cos\alpha)F_i(\beta)}{1-\frac{3}{2i+1}\,\frac{\rho_w}{\rho_m}}\right] \\ &=3h\frac{\rho}{\rho_m}\left(\frac{I}{\pi}-\sin^2\frac{\beta}{2}\right)+3h\frac{\rho}{\rho_m}\sum_{i=1}^{i=\infty}\left[\frac{\frac{3}{2}f_i(\cos\alpha)F_i(\beta)}{(2i+1)\frac{\rho_m}{\rho_w}-3}\right]. \end{aligned} \tag{66}$$

21. The position of any point of the disturbed surface is thus defined by the co-ordinates v and α, v being the elevation or depression of the point relative to the undisturbed spherical surface and α the angular distance of the point from the axis of the disturbing mass. I in (64) and (66) is to be computed from (22) or (23) or their equivalents (24) and (25), according as the point is within or without the perimeter of the disturbing mass. The functions $f_i(\cos\alpha)$ and $F_i(\beta)$ are given by (45) and (47), respectively.

22. The general character of the disturbed surface when the effect of the rearranged water is neglected is evident from (64). It is symmetrical with respect to the axis of the attracting mass. It lies without or within the spherical surface of reference according as I is greater or less than $\pi\sin^2\frac{\beta}{2}$. The values of I for the point of the disturbed surface at the center of the attracting mass, for points along the border of the mass, and for the point 180° from the center of the mass, are given by equations (26), (28), and (30), respectively. If we denote the corresponding values of v by the suffixes 1, 2, 3, we get

$$\begin{aligned} v_1&=3h\frac{\rho}{\rho_m}\left(\sin\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right), \\ &\alpha=0; \\ v_2&=3h\frac{\rho}{\rho_m}\left(\frac{\beta}{\pi}-\sin^2\frac{\beta}{2}\right), \\ &\alpha=\beta; \\ v_3&=3h\frac{\rho}{\rho_m}\left(2\sin^2\frac{\beta}{4}-\sin^2\frac{\beta}{2}\right), \\ &\alpha=\pi. \end{aligned} \tag{67}$$

The meaning of these equations may be most readily understood by reference to Fig. 3. Thus, if the circle FCH represent (in cross-

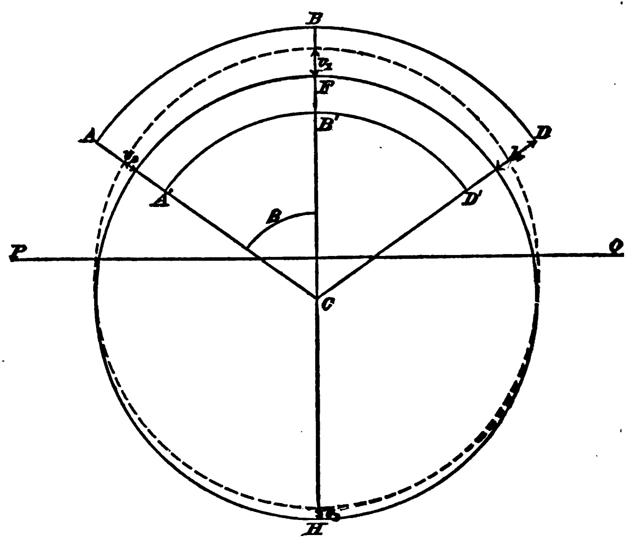

FIG. 3.

section) the undisturbed sea level surface of the earth, and a stratum of matter, as an ice cap, $ABDF$, be added thereto, the new sea-level surface will assume the form indicated by the dotted line. The values of v_1 and v_2, as shown in the diagram, are positive, while the value of v_3 is negative. If, on the other hand, we suppose the space $A'B'D'F$ to be occupied by matter of less density than the average density of the earth's crust, as is the case in a lake basin, the disturbed surface will fall within the undisturbed surface from F to some line PQ, and outside the undisturbed surface from PQ to H, i. e., v_1 and v_2 will be negative and v_3 positive, or, what amounts to the same thing, ρ in (67) will be essentially negative.

23. It is of interest to inquire what angular extent of mass will produce numerical maxima of v_1, v_2, and v_3, supposing the thickness h and the densities ρ and ρ_m constant. By means of the usual criteria it is readily found that

$$\begin{aligned} &v_1 = \text{a maximum for } \beta = 60^\circ, \\ &v_2 = \text{a maximum for } \sin\beta = \frac{2}{\pi}, \text{ or } \beta = 39^\circ\ 32', \\ &v_3 = \text{a maximum for } \beta = 120^\circ. \end{aligned} \tag{68}$$

24. A glance at equation (66) suffices to show that the effect of the free water, if it covers the whole earth, is simply to produce an exaggeration of the type of surface defined by (64) and (65). The series in

the third member of (66) expressing this exaggeration is rapidly converging on account of the diminishing factor

$$\frac{\frac{3}{2}}{(2i+1)\frac{\rho_m}{\rho_w}-3},$$

which is, since $\frac{\rho_m}{\rho_w}$ is about $\frac{11}{2}$,

$$\frac{3}{22i+5}.$$

The essential features of the disturbed surface are, therefore, in any case, defined by (64) or its equivalent (65); and in most cases the effect of the rearranged water may be neglected as unimportant, or as of no greater magnitude than the uncertainties inherent in the data for actual problems.

IX. EVALUATION OF THE DEFINITE INTEGRALS I_1 AND I_2.

25. The equations (67) define the position of the disturbed surface in some of its most characteristic points. To define its position at any other point we must evaluate the elliptic integral I_1 or I_2, which pertains to such point. These integrals have already been expressed [equation (53)] in a series of polar harmonics, which, if more convergent, would suffice for computing I_1 or I_2. It is easy, however, to derive more convergent and convenient series than that of (53), and this is the object of the present Article.

First take I_1 of (24). For brevity put

$$w=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}},$$

and

$$b=\sin\frac{\beta}{2}.$$

Then by Maclaurin's series, or by the binomial theorem, we readily find

$$I_1=2b\int_0^{\frac{\pi}{2}}(1-A\sin^2\gamma_1-B\sin^4\gamma_1-C\sin^6\gamma_1-\ \ .\ \ .\ \ .\ )\,d\gamma_1, \quad (69)$$

in which

$$A=\tfrac{1}{2}w^2(1-b^2),$$
$$B=\tfrac{1}{8}w^4(1+2b^2-3b^4),$$
$$C=\tfrac{1}{16}w^6(1+b^2+3b^4-5b^6),$$
$$.\ \ .\ \ .$$

The even powers of $\sin\gamma_1$ may each be expanded in a series of the form

$$c+d\cos 2\gamma_1+e\cos 4\gamma_1+\ \ .\ \ .\ \ .$$

in which c, d, e, etc., are constants. But since

$$\int_0^{\frac{\pi}{2}}\cos 2n\gamma d\gamma=0,$$

we shall need in these expansions only the values of c. The value of c in the expansion of $(\sin\gamma)^{2n}$ is

$$c=\frac{2n(2n-1)(2n-2)\ \ .\ \ .\ \ .\ \ .\ \ (n+1)}{1\ .\ 2\ .\ 3\ .\ .\ .\ .\ .\ n}\cdot\left(\frac{1}{2}\right)^{2n}$$

Applying this formula, and making the integration in (69), there results

$$I_1=b\pi(1-\tfrac{1}{2}A-\tfrac{3}{8}B-\tfrac{5}{16}C-\ \ .\ \ .\ \ .\ \ ).$$

Hence if we put[1]

$$\begin{aligned}
g_1&=\tfrac{1}{4}(1-b^2),\\
g_2&=\tfrac{3}{64}(1-b^2)(1+3b^2),\\
g_3&=\tfrac{5}{256}(1-b^2)(1+2b^2+5b^4),\\
g_4&=\tfrac{35}{16384}(1-b^2)(5+9b^2+15b^4+35b^6),\\
g_5&=\tfrac{63}{65536}(1-b^2)(7+12b^2+18b^4+28b^6+63b^8),\\
g_6&=\tfrac{231}{1048576}(1-b^2)(21+35b^2+50b^4+70b^6+105b^8+231b^{10}).\\
&.\ \ .\ \ .
\end{aligned}$$

$$\left.\begin{aligned}
&I_1=b\pi(1-g_1w^2-g_2w^4-g_3w^6-\ \ .\ \ .\ .\ \ .\ \ ),\\
&w=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}},\qquad b=\sin\frac{\beta}{2},\qquad \alpha\leqq\beta.
\end{aligned}\right\}\quad(70)$$

This series converges rapidly, except for values of w near unity. In a practical application, to be considered presently, wherein $\beta=38^\circ$ (70) gives, using terms up to that in w^{12}, inclusive, I_1 too great by about 5 per cent. for the case $w=1$. But this is the most unfavorable case,

[1] The general value of g is

$$g_n=\frac{2n(2n-1)(2n-2)\ \ .\ \ .\ \ .\ \ (n+1)}{1^2\ .\ 2^2\ .\ 3^2\ \ .\ \ .\ \ .\ \ n^2\ .\ 2^{3n}}\left\{\begin{array}{l}
+1\ .\ 1\ .\ 3\ .\ 5\ \ .\ \ .\ \ .\ \ (2n-3)\\
+1\ .\ 1\ .\ 3\ .\ 5\ \ .\ \ .\ \ .\ \ (2n-5)1\ .\ nb^2\\
+1\ .\ 1\ .\ 3\ .\ 5\ \ .\ \ .\ \ .\ \ (2n-7)1\ .\ 3\frac{n(n-1)}{1\ .\ 2}b^4\\
+1\ .\ 1\ .\ 3\ .\ 5\ \ .\ \ .\ \ .\ \ (2n-9)\ .\ 1\ .\ 3\ .\ 5\frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}b^6\\
+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\
-1\ .\ 3\ .\ 5\ .\ .\ .\ .\ .\ .\ (2n-1)b^{2n}
\end{array}\right\}$$

and one moreover for which the exact value of I_1 is known from equation (28).

26. By a process entirely similar to that followed above, the expansion of (25) gives, writing for brevity,

$$\nu=\operatorname{cosec}\frac{\alpha}{2},$$

$$I_2=\tfrac{1}{2}b^2\pi\,\nu\left(\begin{array}{l}1+\frac{1}{8}b^2(1+\nu^2)\\ +\frac{1}{64}b^4(3+2\nu^2+3\nu^4)\\ +\frac{5}{1024}b^6(5+3\nu^2+3\nu^4+5\nu^6)\\ +\frac{7}{16384}b^8(35+20\nu^2+18\nu^4+20\nu^6+35\nu^8)\\ +\frac{21}{131072}b^{10}(63+35\nu^2+30\nu^4+30\nu^6+35\nu^8+63\nu^{10})\\ +\ \ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right)^*$$

If in this expression we put

$$k_1=\tfrac{1}{2}+\tfrac{1}{16}b^2+\tfrac{3}{128}b^4+\tfrac{25}{2048}b^6+\tfrac{245}{32768}b^8+\tfrac{1323}{262144}b^{10}+\ \ldots$$
$$k_2=\tfrac{1}{16}+\tfrac{1}{64}b^2+\tfrac{15}{2048}b^4+\tfrac{35}{4096}b^6+\tfrac{735}{262144}b^8+\ \ldots$$
$$k_3=\tfrac{3}{128}+\tfrac{15}{2048}b^2+\tfrac{63}{16384}b^4+\tfrac{315}{131072}b^6+\ \ldots$$
$$k_4=\tfrac{25}{2048}+\tfrac{35}{4096}b^2+\tfrac{315}{131072}b^4+\ \ldots$$
$$k_5=\tfrac{245}{32768}+\tfrac{735}{262144}b^2+\ \ldots$$
$$k_6=\tfrac{1323}{262144}+\ \ldots$$

we find

$$I_2=b\pi(k_1w^{-1}+k_2w^{-3}+k_3w^{-5}+\ \ldots\ ),$$
$$w=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}},\qquad b=\sin\frac{\beta}{2},\qquad \alpha\overline{\gtrless}\beta. \tag{71}$$

This series converges somewhat more rapidly than (70). For the case in which $\beta=38°$ and for the extreme value $w=1$, using terms to that in w^{-11} inclusive, (71) gives I_2 too small by about 3 per cent.

---

*A general expression for the nth term within the brackets, beginning with the third term, for which $n=2$, is the following:

$$\frac{2n(2n-1)(2n-2)\ .\ .\ (n+2)}{1^2\ .\ 2^2\ .\ 3^2\ .\ .\ .\ n^2\ .\ 2^{3n}}\left(\begin{array}{l}+1\ .\ 3\ .\ 5\ .\ .\ .\ (2n-1)\\ +1\ .\ 3\ .\ 5\ .\ .\ .\ (2n-3)\ 1\ .\ n\nu^2\\ +1\ .\ 3\ .\ 5\ .\ .\ .\ (2n-5)\ 1\ .\ 3\ .\ \frac{n(n-1)}{1\ .\ 2}\nu^4\\ +1\ .\ 3\ .\ 5\ .\ .\ .\ (2n-7)\ 1\ .\ 3\ .\ 5\ .\ \frac{n(n-1)(n-2)}{1\ .\ 2\ .\ 3}\nu^6\\ +\ \ldots\ldots\ldots\ldots\\ +1\ .\ 3\ .\ 5\ .\ .\ .\ (2n-1)\nu^{2n}\end{array}\right)b^{2n}$$

27. For points near the border of the disturbing mass I_2 may be expressed by a more rapidly converging series than (71). Thus from equation (23)

$$I_2=\int_0^\beta\left(1-\frac{\cos\beta-\cos\alpha}{\cos p-\cos\alpha}\right)^{\frac{1}{2}}dp.$$

Let

$$\cos\beta-\cos\alpha=2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)=2a.$$

Then

$$I_2=\int_0^\beta\left[1-\frac{a}{(\cos p-\cos\alpha)}-\frac{\frac{a^2}{2}}{(\cos p-\cos\alpha)^2}-\frac{\frac{a^3}{3}}{(\cos p-\cos\alpha)^3}-\ldots\right]dp.$$

Now, if

$$X=\int_0^\beta\frac{dp}{\cos p-\cos\alpha}=\frac{1}{\sin\alpha}\log_e\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)},$$

$$\frac{dX}{d\alpha}=-\sin\alpha\int_0^\beta\frac{dp}{(\cos p-\cos\alpha)^2},$$

$$\frac{d^2X}{d\alpha^2}=-\cos\alpha\int_0^\beta\frac{dp}{(\cos p-\cos\alpha)^2}+2\sin^2\alpha\int_0^\beta\frac{dp}{(\cos p-\cos\alpha)^3},$$

etc.;
whence

$$\int_0^\beta\frac{dp}{(\cos p-\cos\alpha)^2}=-\frac{1}{\sin\alpha}\cdot\frac{dX}{d\alpha},$$

$$\int_0^\beta\frac{dp}{(\cos p-\cos\alpha)^3}=\frac{1}{2\sin^2\alpha}\left(\frac{d^2X}{d\alpha^2}-\cot\alpha\frac{dX}{d\alpha}\right),$$

etc.

The integrals in the third and higher terms of the above series are thus seen to depend on the integral in the second term. Making the requisite differentiations we find, to terms of the third order inclusive,

$$I_2=\beta-\left(\frac{a}{\sin\alpha}+\frac{a^2\cos\alpha}{2\sin^3\alpha}+\frac{a^3(3-2\sin^2\alpha)}{4\sin^5\alpha}+\ldots\right)\log_e\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$-\left(\frac{5a\sin\beta}{16\sin^2\alpha}+\frac{3a^4\cos\alpha\sin\beta}{8\sin^4\alpha}+\ldots\right), \tag{72}$$

$$a=\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

X. SLOPE OF DISTURBED SURFACE.

28. Having derived the requisite formulas for computing the position of any point of the disturbed surface, it remains to determine the slope of this surface relative to the undisturbed surface.

Differentiating equation (64) with respect to α, and dividing the result by the radius of the undisturbed surface r_0, we get

$$\frac{dv}{r_0 d\alpha}=\frac{3h\rho}{r_0\pi\rho_m}\cdot\frac{dI}{d\alpha}. \tag{73}$$

This expresses the slope or inclination of the disturbed to the undisturbed surface in a meridian plane through the center of the disturbing mass; it also expresses the deflection of the plumb line in the same plane.

In order to apply (73) it is essential to have the general value of

$$\frac{dI}{d\alpha}.$$

Since

$$w^n=\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}}\right)^n,$$

$$\frac{dw^n}{d\alpha}=\frac{n}{2}w^n\cot\frac{\alpha}{2},$$

and hence (70) gives

$$\frac{dI_1}{d\alpha}=-\pi\cos\frac{\alpha}{2}\left\{\begin{array}{l}+1g_1w^1\\+2g_2w^3\\+3g_3w^5\\+4g_4w^7\\+\ \cdot\ \cdot\end{array}\right\}. \tag{74}$$

Similarly (71) gives

$$\frac{dI_2}{d\alpha}=-\pi\cos\frac{\alpha}{2}\left\{\begin{array}{l}+\frac{1}{2}K_1w^{-2}\\+\frac{3}{2}K_2w^{-4}\\+\frac{5}{2}K_3w^{-6}\\+\frac{7}{2}K_4w^{-8}\\+\ \cdot\ \cdot\end{array}\right\}. \tag{75}$$

29. Equations (74) and (75) will suffice for the computation of $dI/d\alpha$, except for points near to or at the border of the attracting mass. As α approaches equality to β the above series become less and less convergent, and finally divergent when $\alpha=\beta$ or $w=1$. This may be most

readily seen by differentiating (22) or (23) with respect to α, and then making $\alpha=\beta$. Thus we find

$$\frac{dI}{d\alpha}=-\frac{1}{2}\sin\beta\int_0^\beta\frac{dp}{\cos\beta-\cos p}=-\frac{1}{2}\left[\log_e\frac{\sin\frac{\beta+p}{2}}{\sin\frac{\beta-p}{2}}\right]_0^\beta=-\infty.$$

Likewise the integrals (24) and (25) become, after differentiating them with respect to α and then making $\alpha=\beta$,

$$\frac{dI_1}{d\alpha}=-\int_0^{\frac{\pi}{2}}\frac{\sec^2\gamma_1\tan^2\gamma_1 d\gamma_1}{\left(\sec^2\frac{\beta}{2}+\tan^2\gamma_1\right)^{\frac{3}{2}}},\tag{A}$$

$$\frac{dI_2}{d\alpha}=-\int_0^{\frac{\pi}{2}}\frac{\sec^2\gamma_2 d\gamma_2}{\left(\sec^2\frac{\beta}{2}+\tan^2\gamma_2\right)^{\frac{1}{2}}}\tag{B}$$

$$=-\int_0^{\frac{\pi}{2}}\frac{\sec^2\frac{\beta}{2}\sec^2\gamma_2 d\gamma_2}{\left(\sec^2\frac{\beta}{2}+\tan^2\gamma_2\right)^{\frac{3}{2}}}-\int_0^{\frac{\pi}{2}}\frac{\sec^2\gamma_2\tan^2\gamma_2 d\gamma_2}{\left(\sec^2\frac{\beta}{2}+\tan^2\gamma_2\right)^{\frac{3}{2}}}$$

$$=-1+\frac{dI_1}{d\alpha}.$$

This shows the equality of (A) and (B) since (B) is plainly infinite, its value being

$$-\left[\log_e\frac{\tan\gamma_2+\left(\sec^2\frac{\beta}{2}+\tan^2\gamma_2\right)^{\frac{1}{2}}}{\sec^2\frac{\beta}{2}}\right]_0^{\frac{\pi}{2}}.$$

30. This failure of equations (74) and (75) for points at the border of the attracting mass arises from the fact that the expressions (20) and (21), though very approximate for the magnitude of the potential V, are not sufficiently general to give an accurate value of $dV/d\alpha$, or the attraction in the direction of the arc α for those points. To determine the slope of the disturbed surface at the immediate border of the disturbing mass a special investigation is requisite.

Since by equations (3) and (6) the slope is expressed by

$$\frac{dv}{r_0 d\alpha}=\frac{3}{4}\cdot\frac{1}{r_0\pi\rho_m}\cdot\frac{dV}{r_0 d\alpha},\tag{76}$$

we may derive an expression for the attraction $dV/r_0 d\alpha$ directly. The exact expression for the horizontal attraction towards the axis of the mass of any element mass is, using the same notation as in Article III,

$$-\rho\frac{4r^3dr\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}\,d\theta\cos\lambda d\lambda}{\left[(r-r')^2+4rr'\sin^2\frac{\theta}{2}\right]^{\frac{3}{2}}},$$

and the integral of this is $dV/r_0 d\alpha$.

Now, as heretofore, let

$$r=r_0+u, \qquad r'=r_0+v.$$

In addition put

$$\eta=r-r',$$

so that

$$d\eta=dr,$$
$$\eta=-v \text{ for } r=r_0,$$
$$\eta=h-v \text{ for } r=r_0+h.$$

Also let

$$\xi=2r_0\sin\frac{\theta}{2},$$

whence

$$d\xi=r_0\cos\frac{\theta}{2}\,d\theta,$$

$$\cos\frac{\theta}{2}=\sqrt{1-\left(\frac{\xi}{2r_0}\right)^2}.$$

Making these substitutions and neglecting terms of the order

$$\frac{u}{r_0},\ \frac{v}{r_0},\ \text{and}\ \left(\frac{\xi}{2r_0}\right)^2,$$

the above expression becomes

$$\rho\frac{\xi^2d\xi d\eta\cos\lambda d\lambda}{(\xi^2+\eta^2)^{\frac{3}{2}}}.$$

Integrating with respect to η, and substituting the limits given above, there results

$$\left[\frac{(h-v)d\xi}{[\xi^2+(h-v)^2]^{\frac{1}{2}}}+\frac{vd\xi}{(\xi^2+v^2)^{\frac{1}{2}}}\right]\cos\lambda d\lambda.$$

If, now, we suppose the attracted point on the border of the attracting mass, the limits of ξ will be 0, and, with sufficient approximation, $2r_0 \sin\beta \cos\lambda = c\cos\lambda$, say. Integrating with respect to ξ, and substituting these limits, we get

$$\rho(h-v)\cos\lambda d\lambda \log_e\left(\sqrt{1+\frac{c^2\cos^2\lambda}{(h-v)^2}}+\frac{c\cos\lambda}{h-v}\right)$$

$$+\rho v\cos\lambda d\lambda \log_e\left(\sqrt{1+\frac{c^2\cos^2\lambda}{v^2}}+\frac{c\cos\lambda}{v}\right).$$

It remains to integrate these last expressions with respect to λ between the limits 0 and $\frac{\pi}{2}$. An application of the formula for integration by parts will readily transform them to elliptics, but since their element functions decrease very rapidly from the lower to the upper limit, the following process[1] will suffice. Consider the integral

$$\int_0^\lambda \cos\lambda d\lambda \log_e\left(\sqrt{1+\frac{c^2\cos^2\lambda}{v^2}}+\frac{c\cos\lambda}{v}\right),$$

in which λ is such that $\left(\frac{v}{c\cos\lambda}\right)^2$ may be neglected in comparison with unity. In the cases we have to consider $\left(\frac{v}{c\cos\lambda}\right)^2$ will not exceed $\frac{1}{100}$ if $\cos\lambda=\frac{1}{100}$ or $\lambda=89^\circ\ 25'$ about. Then, since

$$\log_e\left[\sqrt{1+\frac{c^2\cos^2\lambda}{v^2}}+\frac{c\cos\lambda}{v}\right]$$

$$=\log_e\left\{\frac{2c\cos\lambda}{v}\left[1+\frac{1}{4}\left(\frac{v}{c\cos\lambda}\right)^2+\ \ .\ \ .\ \ .\right]\right\},$$

the above integral becomes

$$\int_0^\lambda \cos\lambda d\lambda \log_e\frac{2c\cos\lambda}{v}=\log_e\frac{2c}{v}\int_0^\lambda\cos\lambda d\lambda+\int_0^\lambda\cos\lambda d\lambda\log_e\cos\lambda$$

$$=\left(\log_e\frac{2c}{v}-1\right)\sin\lambda+\log_e(1+\sin\lambda)+(\sin\lambda-1)\log_e\cos\lambda.$$

But since $\sin\lambda$ is very nearly unity, the last expression reduces to

$$\log_e\left(\frac{4c}{v}\right)-1.$$

The error of this integral arising from the use of λ instead of $\frac{\pi}{2}$ as the upper limit is less than

$$\left(\frac{\pi}{2}-\lambda\right)\cos\lambda\log_e\left\{\sqrt{1+\frac{c^2\cos^2\lambda}{v^2}}+\frac{c\cos\lambda}{v}\right\},$$

which, if $\cos\lambda=\frac{1}{100}$ and $c\cos\lambda/v=10$, amounts to about $\frac{1}{3000}$.

[1] Given in a somewhat different form by Helmert in Theorieen der höheren Geodäsie, Vol. II, p. 322.

For the entire attraction, therefore, of the mass for a point on its border we get

$$\frac{dV}{r_0 d\alpha} = -2\rho \left\{ (h-v)\left(\log_e \frac{4c}{h-v} - 1\right) + v\left(\log_e \frac{4c}{v} - 1\right) \right\}$$

$$= -2\rho \left\{ h\left(\log_e \frac{4c}{h-v} - 1\right) + v \log_e \frac{h-v}{v} \right\}.$$

Finally, restoring in this last expression the value of c, viz, $c = 2r_0 \sin\beta$, (76) becomes

$$\frac{dv}{r_0 d\alpha} = -\frac{3}{2}\frac{h\rho}{r_0 \pi \rho_m}\left\{ \log_e \frac{8r_0 \sin\beta}{h-v} + \frac{v}{h}\log_e \frac{h-v}{v} - 1 \right\}. \tag{77}$$

XI. DISTURBED CENTER OF GRAVITY OF EARTH.

31. Thus far the disturbed surface has been referred to a spherical surface concentric with the earth's center of gravity before the disturbance arose. In determining the effects of the ice mass in glacial times this is the proper surface of reference, since we wish to know the distortion of the sea level in those times relative to the sea level in preceding and following epochs. If, however, it is desired to consider the joint effect in distorting the sea level of existing masses, like the continents, on the hypothesis that such masses rest on the surface of a centrobaric sphere, a better surface of reference will obviously be the disturbed or existing center of gravity of the earth. The use of the latter center will require a slight modification of the preceding formulas defining the disturbed sea surface.

To determine the radial displacement of the earth's center of gravity due to the addition of such a superficial mass as we have considered, it is only necessary to equate the statical moment of that mass to the statical moment of the earth's mass, the moment plane being perpendicular to the axis of the disturbing mass at the undisturbed center of gravity of the earth. The moment of an elementary ring of angular radius β', measured from the axis of the disturbing mass, is to our order of approximation

$$2r_0^3 h\rho\pi \sin\beta' \cos\beta' d\beta'.$$

Hence, if σ denote the displacement sought and M the earth's mass,

$$M\sigma = r_0^3 h\rho\pi \int_0^\beta 2 \sin\beta' \cos\beta' d\beta'$$

$$= r_0^3 h\rho\pi \sin^2\beta.$$

Therefore, by substitution of the value of M given in equation (3), we find

$$\sigma = \frac{3}{4} h \frac{\rho}{\rho_m} \sin^2\beta. \tag{78}$$

Now, the elevation of any point of the disturbed surface relative to the sphere in the new position will be less than its elevation relative to the sphere in the former position by an amount whose value to the proper degree of approximation is

$$\sigma \cos \alpha,$$

α being, as heretofore, the angular distance of the point from the axis of the disturbing mass. That is, if v' denote what v becomes by the change in position at the sphere of reference,

$$v'=v-\sigma \cos \alpha.$$

Hence, by virtue of (64) and (78) we find for the equation of the disturbed surface when the sphere of reference is concentric with the disturbed center of gravity of the earth,

$$v'=3h\frac{\rho}{\rho_m}\left[\frac{I}{\pi}-\sin^2\frac{\beta}{2}\left(1+\cos\alpha\cos^2\frac{\beta}{2}\right)\right]. \tag{79}$$

And the inclination of the disturbed surface to the surface of reference is

$$\frac{dv'}{r_0 d\alpha}=3h\frac{\rho}{r_0\pi\rho_m}\left(\frac{dI}{d\alpha}+\frac{\pi}{4}\sin\alpha\sin^2\beta\right). \tag{80}$$

XII. EQUATIONS OF DISTURBED SURFACE WHEN THE DISTURBING MASS IS OF VARIABLE THICKNESS.

32. Throughout the preceding investigations the thickness of the attracting mass has been considered uniform. On this account the range of application of the formulas derived is somewhat narrow. It may be remarked, however, before proceeding to extend the investigation to more complex masses, that inasmuch as the data for actual problems will be in general more or less uncertain, or to a large extent ideal, formulas of a more comprehensive and hence more complex character are not specially desirable. Approximate calculations of a rather rough sort in some cases will be as good as the data for those calculations. The effects assigned by the foregoing equations will be for the most part in excess of the probable actual effects, and in so far as computation can contribute arguments pertinent to observed facts the maximum effects will be most essential. On the other hand it will be desirable in some cases to get an idea of the inferior limiting effects. The most important of these cases relates to the extent of submergence attributable to the ice cap of the glacial epoch. There would seem to be little probability of uniform thickness in such a cap. Apparently

some sort of regular decrease in thickness (with here and there considerable though comparatively unimportant deviations) from the center to the perimeter of the mass is more probable. Such a law of decrease is expressed by the equation

$$\varphi(\beta)=h=h_0\left[1-\left(\frac{\sin\frac{\beta}{2}}{\sin\frac{\beta_0}{2}}\right)^n\right], \tag{81}$$

in which h is the thickness along any radial line whose angular distance from the axis of the mass is β, h_0 is the thickness along the axis, and β_0 is the angular radius of the perimeter of the mass. In brief, h is a function of β, as stated by the first member of the equation. The exponent n must be a positive number and may be for our purposes restricted to integer values. In order to determine the effects of masses conforming to the above, and, in general, any law requiring symmetry of mass with respect to a radial axis, we shall devote the present Article to the necessary extension of the formulas already derived.

33. The differential of equation (64) with respect to β gives

$$\frac{dv}{d\beta}d\beta=3\frac{h\rho}{\pi\rho_m}\left(\frac{dI}{d\beta}d\beta-\pi\frac{d\sin^2\frac{\beta}{2}}{d\beta}d\beta\right). \tag{82}$$

This expresses the elevation of the disturbed surface due to an annulus of angular radius β, of angular width $d\beta$, and height h, the density ρ being uniform. If in this equation we make h a function of β, or write $h=\varphi(\beta)$, and integrate between the proper limits, the result will be the elevation of the disturbed surface due to a mass whose thickness conforms to the law expressed by $\varphi(\beta)$. Calling, for the sake of distinction, v'' the new value of the elevation of the disturbed surface, and the proper limits of β, β_1, and β_2, the result of this integration is

$$v''=\frac{3\rho}{\pi\rho_m}\int_{\beta_1}^{\beta_2}\frac{d\left(I-\pi\sin^2\frac{\beta}{2}\right)}{d\beta}\varphi(\beta)d\beta. \tag{83}$$

This equation assigns the effect of any homogeneous mass whose bounding surface is one of revolution about a radial axis, subject to the restriction that the maximum thickness of the mass may be neglected in comparison with the radius of the earth. It is obvious, however, that the integral in (83) may be impracticably complex for some forms of $\varphi(\beta)$. To avoid undue complexity and at the same time attain results suitable for our special purposes we shall here confine

attention to that form of $\varphi(\beta)$ expressed by equation (81). For this function we have, considering the whole mass,

$$\beta_1=0, \qquad \beta_2=\beta_0,$$
$$\varphi(\beta)=0 \text{ for } \beta=\beta_0,$$

$$d\varphi(\beta)=-n\frac{\left(\sin\frac{\beta}{2}\right)^{n-1}d\left(\sin\frac{\beta}{2}\right)}{\sin^n\frac{\beta_0}{2}}.$$

Then, observing that $I=0$ for $\beta=0$, (83) becomes

$$v''=3\frac{h_0\rho}{\pi\rho_m}\left[\frac{n}{\sin^n\frac{\beta_0}{2}}\int_0^{\beta_0}I\left(\sin\frac{\beta}{2}\right)^{n-1}d\left(\sin\frac{\beta}{2}\right)-\frac{n}{n+2}\pi\sin^2\frac{\beta_0}{2}\right]. \qquad (84)$$

The definite integral in this expression depends on and will in general be no less complex than I, which is defined by (22) to (25). An examination of (24) and (25) shows that for points of the disturbed surface within the perimeter of the disturbing mass

$$\int_0^{\beta_0}I\left(\sin\frac{\beta}{2}\right)^{n-1}d\sin\frac{\beta}{2}=\int_0^{\alpha}I_2\left(\sin\frac{\beta}{2}\right)^{n-1}d\sin\frac{\beta}{2}$$
$$+\int_\alpha^{\beta_0}I_1\left(\sin\frac{\beta}{2}\right)^{n-1}d\sin\frac{\beta}{2}. \qquad (85)$$

For points of the disturbed surface without the perimeter of the mass it is only necessary to replace I in (84) by I_2 of (25), or replace the limit α in (85) by β_0. By means of the series (70) and (71), or the harmonic series (53), (84) may be evaluated for any point of the disturbed surface.

For two points of the disturbed surface, namely, at the center of the mass and 180° from that center, (84) yields to direct integration. The process of evaluation is as follows:

The integral in (84) is a function of α, β_0, and n. Let it be symbolized by $f(\alpha\beta_0 n)$. For the two points noted above let this function be distinguished by the suffixes 1 and 2, respectively, so that it becomes

$$f_1(\alpha, \beta_0, n) \text{ for } \alpha=0,$$

and

$$f_2(\alpha, \beta_0, n) \text{ for } \alpha=\pi.$$

Then, since from (26), $I=\pi\sin\frac{\beta}{2}$ for $\alpha=0$,

$$f_1(\alpha, \beta_0, n)=\frac{n}{n+1}\pi\sin\frac{\beta_0}{2}. \qquad (86)$$

Likewise, since from (30), $I=2\pi\sin^2\frac{\beta}{4}$ for $\alpha=\pi$,

$$f_2(\alpha,\beta_0,n)=2\pi\sin^2\frac{\beta_0}{4}-2\pi\int_0^{\beta_0}\left(\frac{\sin\frac{\beta}{2}}{\sin\frac{\beta_0}{2}}\right)^n d\sin^2\frac{\beta}{4}. \qquad (87)$$

$$=\frac{\pi}{4}\left(4-2\cos\frac{\beta_0}{2}-\beta_0\operatorname{cosec}\frac{\beta_0}{2}\right)\text{ for } n=1,$$

$$=\frac{\pi}{3}\left(4\cos^2\frac{\beta_0}{4}-1\right)\tan^2\frac{\beta_0}{4}\qquad\text{for } n=2,$$

. . .

When n is more than a few units the integral in (87) may be evaluated by rapidly converging series. Thus, call the required integral B, and let

$$\sin\frac{\beta}{2}=\varkappa\sin\frac{\beta_0}{2}.$$

Then

$$B=\int_0^{\beta_0}\left(\frac{\sin\frac{\beta}{2}}{\sin\frac{\beta_0}{2}}\right)^n d\sin^2\frac{\beta}{4}$$

$$=\frac{1}{2}\sin^2\frac{\beta_0}{2}\int_0^1\frac{\varkappa^{n+1}d\varkappa}{\sqrt{1-\varkappa^2\sin^2\frac{\beta_0}{2}}} \qquad (88)$$

$$=\frac{1}{2}\left(\frac{\sin^2\frac{\beta_0}{2}}{n+2}+\frac{1}{2}\frac{\sin^4\frac{\beta_0}{2}}{n+4}+\frac{3}{8}\frac{\sin^6\frac{\beta_0}{2}}{n+6}+\ldots\right).$$

34. It will be particularly essential for our purposes to evaluate (84) for points outside the border of the disturbing mass. The integral required is, if we write for brevity

$$b=\sin\frac{\beta}{2}\text{ and } b_0=\sin\frac{\beta_0}{2},$$

$$nb_0^{\ -n}\int_0^{b_0}I_2b^{n-1}db.$$

Now, I_2 from the equation preceding (71), may be written thus:

$$I_2=\pi(j_1b^2+j_2b^4+j_3b^6+\ldots),$$

(139)

in which j_1, j_2, etc., are obvious functions of α or $\nu = \operatorname{cosec} \alpha$. Therefore the above integral becomes

$$nb_0^{-n}\int_0^{b_0} I_2 b^{n-1} db = \pi \left\{ \begin{array}{l} +\frac{n}{n+2} j_1 b_0^2 \\ +\frac{n}{n+4} j_2 b_0^4 \\ +\frac{n}{n+6} j_3 b_0^6 \\ + \ . \ \ . \ \ . \end{array} \right\} \tag{89}$$

$$= \pi S, \text{ say.}$$

35. Let the value of S in the last equation for points at the border of the disturbing mass where $\alpha = \beta_0$ be denoted by S_1. Also denote by v_1'', v_2'', and v_3'' the elevations of the disturbed surface at the center of the mass, at its border, and 180° from its center. Then equation (84), by means of the results in (86) to (89), gives the following equations analogous to the group (67):

$$v_1'' = 3h_0 \frac{\rho}{\rho_m}\left(\frac{n}{n+1} \sin \frac{\beta_0}{2} - \frac{n}{n+2} \sin^2 \frac{\beta_0}{2}\right),$$

$$\alpha = 0;$$

$$v_2'' = 3h_0 \frac{\rho}{\rho_m}\left(S_1 - \frac{n}{n+2} \sin^2 \frac{\beta_0}{2}\right), \tag{90}$$

$$\alpha = \beta_0;$$

$$v_3'' = 3h_0 \frac{\rho}{\rho_m}\left(2 \sin^2 \frac{\beta_0}{4} - 2B - \frac{n}{n+2} \sin^2 \frac{\beta_0}{2}\right),$$

$$\alpha = \pi.$$

36. To define the slope of the disturbed surface it is in general necessary to differentiate (84) with respect to α. The result is of a complex character and subject to discontinuity for points at the border of the mass. For practical purposes, however, it will suffice to make use of $\Delta v''/\Delta\alpha$ instead of the differential coefficient, and thus determine average slopes over some finite portion of a meridian section of the disturbed surface.

37. In discussing the disturbance of the sea level attributable to the ice mass of the glacial epoch, it will be of interest to estimate the effect of the rearranged free water. For this purpose we may extend the second term of the third member of equation (66) so as to make it assign the effect of the rearrânged water when the mass is of variable as well as uniform thickness. The process is strictly analogous to that

followed in deriving (83) and (84) from (64). Thus, confining attention to the form of mass defined by (81), and writing, as in section **34**,

$$\varphi(\beta)=h_0\left[1-\left(\frac{\sin\frac{\beta}{2}}{\sin\frac{\beta_0}{2}}\right)^n\right]$$

$$=h_0(1-b_0^{-n}b^n),$$

and

$$F_i(\beta)=F_i(b),$$

we readily find from (66)

$$\Delta v''=\frac{9}{2}h_0\frac{\rho}{\rho_m}\sum_{i=1}^{i=\infty}\left(\frac{f_i(\cos\alpha)nb_0^{-n}\int_0^{b_0}F_i(b)b^{n-1}db}{(2i+1)\frac{\rho_m}{\rho_w}-3}\right). \qquad (91)$$

A few values of $F_i(b)$ derived from (45) and (47) are the following:

$$F_1(b)=2(b^2-b^4),$$
$$F_2(b)=2(b^2-3b^4+2b^6),$$
$$F_3(b)=2(b^2-6b^4+10b^6-5b^8),$$
$$F_4(b)=2(b^2-10b^4+30b^6-35b^8+14b^{10}),$$
$$F_5(b)=2(b^2-15b^4+70b^6-140b^8+126b^{10}-42b^{12}).$$

From these the corresponding integrals

$$nb_0^{-n}\int_0^{b_0}F_i(b)b^{n-1}db$$

can be readily derived. Thus, for example,

$$nb_0^{-n}\int_0^{b_0}F_5(b)b^{n-1}db=2n\left\{\begin{array}{l}+\frac{1}{n+2}b_0^2\\ -\frac{15}{n+4}b_0^4\\ +\frac{70}{n+6}b_0^6\\ -\frac{140}{n+8}b_0^8\\ +\frac{126}{n+10}b_0^{10}\\ -\frac{42}{n+12}b_0^{12}\end{array}\right\}. \qquad (92)$$

38. A common property of the formulas (84) to (92), both inclusive, is worthy of notice. They all refer to a mass whose thickness conforms to the law (81), namely,

$$\varphi(\beta)=h=h_0\left(1-\frac{\sin^n\frac{\beta}{2}}{\sin^n\frac{\beta_0}{2}}\right).$$

When $n=\infty$ this gives $\varphi(\beta)=h=h_0$, or the thickness of the mass is uniform. Therefore the formulas (84) to (92) should return to the forms applicable to a mass of uniform thickness on making $n=\infty$. Such is the case. Thus (84) becomes (64), as is readily seen by an application of the formula for integration by parts. Likewise, for $n=\infty$ equations (86), (87), and (89) become (26), (30), and (28), respectively, and the group (90) assumes the simpler forms of the group (67).

B. APPLICATIONS.

XIII. RELATIVE POSITIONS OF LEVEL OR EQUIPOTENTIAL SURFACES IN A LAKE BASIN.

39. Consider the question stated in section **2**, (*a*), relative to the level surfaces in a lake basin. In this case it is required to determine the difference in elevation at the center of the basin of two level surfaces which intersect along its perimeter. The first two of equations (90) give

$$v_2''-v_1''=3h_0\frac{\rho}{\rho_m}\left(S_1-\frac{n}{n+1}\sin\frac{\beta_0}{2}\right). \tag{93}$$

This represents the difference in elevation of a level or liquid surface at the center and at the border of the basin. ρ must be understood as the excess or defect in density of the liquid relative to the average density of the superficial strata of the earth. Thus, if the liquid in question be water,

$$\rho=-(\tfrac{1}{2}\rho_m-1)=-1.8, \text{ approximately.}$$

If we differentiate (93), regarding ρ as variable, the result is

$$\varDelta(v_2''-v_1'')=3h_0\frac{\varDelta\rho}{\rho_m}\left(S_1-\frac{n}{n+1}\sin\frac{\beta_0}{2}\right). \tag{94}$$

This expresses the required separation of the two level surfaces in question; i. e., the separation at the center of the basin of two level surfaces which intersect at the border and which are the free surfaces of two liquids whose difference in density is $\varDelta\rho$.

To illustrate more fully the meaning of (93) and (94) let $ABDE$ in Fig. 4 represent a cross section through the axis of the basin. ACB is a circular arc parallel to the section of the sphere of reference. If the basin be filled with water the section of the water surface will lie below ACB as $AC'B$. If the water be removed the corresponding sec-

tion of the level surface through A and B will fall below $AC'B$ as $AC''B$. Hence, in the diagram

$$v_2''-v_1''=CC', \qquad \Delta(v_2''-v_1'')=C'C''.$$

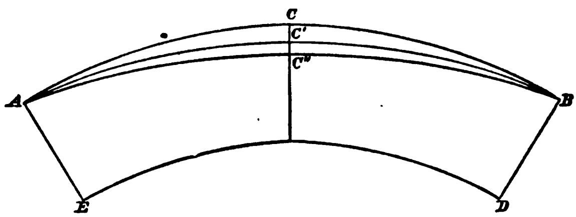

FIG. 4.

40. To get some numerical values, let us assume the following data:

$$h_0=1{,}000 \text{ feet}, \qquad \beta_0=\text{arc of } 1^\circ, \qquad \rho_m=5{\cdot}5,$$
$$\rho=-1.8, \qquad \Delta\rho=1,$$

this latter being approximate difference in density of air and water. Then

$$3h_0\frac{\rho}{\rho_m}=982 \text{ feet},$$

$$3h_0\frac{\Delta\rho}{\rho_m}=545 \text{ feet}.$$

The assumed value of β_0 is equivalent to about 69 miles measured along the earth's surface.

Now, by means of equations (86) to (89), we find the following table of results corresponding to several values of n, which defines the shape of the basin—see equation (81). The results in the fifth column of the table express the difference in elevation of a water surface at the center and at the border of the basin; and those in the sixth column express the depression of the level surface at the center of the basin consequent upon substituting air for water as the attracting mass.

Table of values showing relative positions of level surfaces in a lake basin 140 *miles in diameter and of* 1,000 *feet maximum (axial) depth.*

n	$\frac{n}{n+1}\sin\frac{\beta_0}{2}$	S_1	$\frac{n}{n+1}\sin\frac{\beta_0}{2}-S_1$	$v_2''-v_1''$	$\Delta(v_2''-v_1'')$
				Feet.	*Feet.*
1	0.00436	0.00161	0.00275	2.70	1.50
2	582	245	337	3.31	1.84
3	654	298	356	3.50	1.94
4	698	333	365	3.58	1.99
5	727	359	368	3.61	2.01
6	748	379	369	3.62	2.01
7	764	395	369	3.62	2.01
8	776	407	369	3.62	2.01
9	786	418	368	3.61	2.01
10	793	427	366	3.59	1.99
∞	0.00873	0.00556	0.00317	3.11	1.73

It will be observed that the differences in the last two columns of the above table rise to a maximum between the arguments $n=5$ and $n=9$. That they should do so is evident from an inspection of equations (89), (93), and (94).

The deflection of the plumb line toward the land along the border of such a lake, supposing it of uniform depth or $n=\infty$, would be by equation (77) 11″ or 16″, according as the basin is filled with water or air, and the slope of the water surface at the immediate border would be 0.28 feet per mile.

41. An interesting inference, which might be drawn from the solution of the above problem, is that in triangulating a large lake we should expect to see from shore to shore with somewhat less elevations of the points of observation than the usual formulas for intervisibility of points on the earth's surface would require. This would be a correct inference, however, only in case the defect in potential due to the water in the lake basin is not offset by an excess in potential due to some local or general distribution of matter within the earth's crust.

XIV. VARIATIONS OF SEA LEVEL ATTRIBUTABLE TO CONTINENTAL GLACIERS OR ICE CAPS.

42. As a second application of the preceding theory we shall investigate the attractive effects of the ice mass of the glacial epoch, assuming that the earth's crust did not yield to the pressure of the ice. This is the problem of section 2 (*b*).

This problem in its physical aspects presents two difficulties, the first of which has not been alluded to in the foregoing sections, and the second only partially considered.

The first of these difficulties is to account for the enormous quantity of water required to form such an ice mass as is supposed to have covered our northern hemisphere during the glacial period. This mass has been usually estimated as not less than 5,000 feet thick at its center, and to have extended 30° to 90° from that center. It is generally assumed to have diminished in thickness with some approximation to regularity from the center to the perimeter. The superior limit for this shape of mass would be a sheet of uniform thickness, and the inferior limit a meniscus increasing slowly in thickness from its perimeter towards its axis. Evidently, if the water forming such a mass were drawn from the ocean, the latter would undergo a considerable diminution in elevation, and this diminution might nearly counterbalance the attractive effects of the ice in elevating the water along its border. The view advanced by Dr. Croll, however, assumes that there is an alternation of glaciation at the poles, the epoch of minimum ice cap at the one corresponding to the epoch of maximum ice cap at the other. This, granting the sufficiency of ice in the two caps, would make the quantity of free water in the ocean substantially constant. In the absence of definite information on this point, it must be admitted that considerable

uncertainty may properly be attributed to our computed variations of sea level, although the equivalent lowering of the sea, if the water in the ice is drawn therefrom, will be determined for each assumed mass.

43. The second difficulty arises from the fact that such a large mass as we seem compelled to assume for the ice cap would produce an extensive rearrangement of the sea water; and we ought therefore, in computing the potential at any point of the disturbed surface, to take account of this rearrangement. We have shown how to do so in an ideal case, which presents effects for the elevation or depression and slope of the disturbed surface greater than the probable actual effects. The actual effects, as we shall indicate, probably lie about midway between those assigned by the formulas for the ideal case and those assigned by the formulas which neglect the potential due to the rearranged free water. But on account of the difficulty in fixing an exact limit for the actual effects, our computed results will be subject to a small range of uncertainty, which may be regarded, however, as no greater than the inherent uncertainty in the more important data of the problem.

44. Let us now take for the mean density of the earth ρ_m, for the density of ice ρ, for the thickness of the ice along its axis h_0, and for the angular extent of the mass β_0, the following values:

$$\rho_m = 5.5, \qquad h_0 = 10{,}000 \text{ feet},$$

$$\rho = 1,\text{[1]} \qquad \beta_0 = \text{arc of } 38^\circ.$$

This value of β_0 corresponds to about 2,600 miles measured along the earth's surface; it is also very nearly that angular extent of mass, of uniform thickness, which produces the maximum upheaval of water along its border. (See section **23**.)[2]

45. As we shall compute the effects of masses corresponding to several values of the index n, equation (81), it will be of interest to define with some precision the shape of the exterior surface of each mass. To do this it will suffice to give the slope of the surface of any mass at

[1] This value for the density of ice is about 8 per cent. too great; but by using it in the formulas which do not take account of the potential due to the rearranged free water, we shall get results differing only slightly from the probable results.

[2] If the water in the ice cap were drawn wholly from the free sea water the angular radius of a mass producing maximum upheaval along its border would be much less than the value given by the second of equation (68), for in this case the rise in sea level due to the attraction of the cap would be offset partly by the fall due to the withdrawal of the water. The difference between the rise and fall just mentioned is v_2'' of (90), minus r of (96), or

$$3h_0 \frac{\rho}{\rho_m}\left(S_1 - \frac{n}{n+2}\sin^2\frac{\beta_0}{2}\right) - \frac{n}{n+2}\, h_0 \sin^2\frac{\beta_0}{2}.$$

The value of β_0 which will render this difference a maximum is easily found, but the result is of little interest since the corresponding mass would produce effects much smaller than the possible effects of continental glaciers.

several points in a meridian plane. The differential of equation (81) gives

$$\frac{dh}{r_0 d\beta} = -\frac{nh_0}{2r_0}\left(\frac{\sin\frac{\beta}{2}}{\sin\frac{\beta_0}{2}}\right)^n \cot\frac{\beta}{2}. \qquad (95)$$

This expresses the slope or inclination in a meridian plane of the bounding surface of the attracting mass to the spherical surface of reference. Using the above values of h_0 and β_0, and for the radius of the earth (see section 4) log r_0=7.32020, the following table of values has been computed. The slopes are expressed in feet per mile. Several of the curves whose slopes are given are delineated (with greatly exaggerated radial scale) in Fig. 5.

Table showing meridian slopes of exterior bounding surfaces of assumed ice masses. [*See equations* (81) *and* (95).]

n	Slopes in feet per mile corresponding to β=								
	0°	5°	10°	15°	20°	25°	30°	35°	38°
	Feet.	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*	*Feet.*
1	3.88	3.88	3.86	3.85	3.82	3.78	3.75	3.70	3.67
2	0.00	1.04	2.07	3.08	4.07	5.04	5.96	6.84	7.34
3	.00	.21	.83	1.85	3.26	5.02	7.10	9.47	11.00
4	.00	.03	.29	.99	2.32	4.45	7.53	11.66	14.67
5	.00	.01	.10	.49	1.55	3.70	7.48	13.46	18.34
6	.00	.00	.03	.24	.99	2.95	7.14	14.92	22.01
7	.00	.00	.01	.11	.62	2.29	6.62	16.08	25.68
8	.00	.00	.00	.05	.47	1.73	6.02	16.97	29.34
9	.00	.00	.00	.02	.23	1.30	5.38	17.64	33.01
10	.00	.00	.00	.01	.13	.96	4.75	18.10	36.68
∞	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	∞

The numbers in the above table show that for n=1 the slope of the bounding surface, as defined by equation (81), is steepest at the axis of the mass and decreases slowly from the axis towards the border. For values of n greater than unity the bounding surfaces slope up with decreasing rapidity from the border to the axis of the mass, the amount of slope diminishing to zero at the axis in each case. The features here enumerated will hold for any extent of mass, i. e., for any value of β_0. It will be observed also that the slope for any value of n is directly proportional to the axial thickness h_0. Hence the slopes corresponding to any other thickness than that assumed (10,000 feet) may be readily computed from the table.

46. As to the actual form of the bounding surface of the ice mass of the glacial epoch we have no precise information. It is generally

assumed, however, that the mass was thickest along its axis, and that the thickness decreased with some approach to regularity between the axis and the border.[1] The slope of ascent along the border has been estimated as 10 to 35 feet per mile.[2] Nordenskjöld observed a rise of 7,000 feet in 280 miles from the border of the ice plains of Greenland.[3]

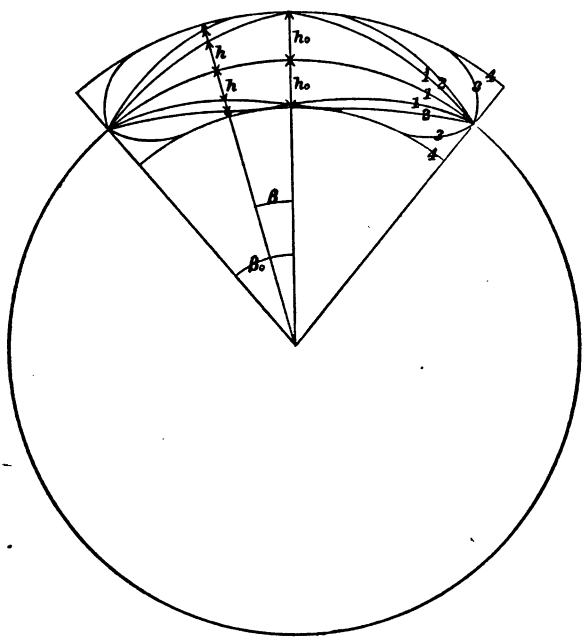

FIG. 5.

Scale for section of sphere $= \frac{1}{84000000}$,
Scale for disturbing mass $= \frac{1}{240000} = \frac{350}{84000000}$,

$$h = h_0 \left\{ 1 - \frac{\sin^n \frac{1}{2}\beta}{\sin^n \frac{1}{2}\beta_0} \right\}, \qquad h_0 = 10,000 \text{ feet.}$$

For curve 1, $n=1$; for curve 2, $n=2$; for curve 3, $n=10$; for curve 4, $n=\infty$.

This corresponds to an average slope of 25 feet per mile. It seems most probable that the slope of such a mass would be steepest at its border and diminish gradually towards its center. Whatever may have been the actual slopes it is thought that the preceding table affords a sufficiently comprehensive variety. Our equations, it is true, will assign the effects of an indefinite variety of other forms, but in the absence of more complete actual data the simple forms whose slopes have been computed are considered adequate.

[1] The directly opposite view is maintained by W J McGee. See his paper on Maximum Synchronous Glaciation, Proceedings of the American Association for the Advancement of Science, Vol. xxix, 1880.

[2] Croll, Climate and Cosmology, p. 244.

[3] *Ibid.*, p. 245

47. In order to form an idea of the amount of water congealed in the masses whose shapes have just been defined we will compute the thicknesses of spherical shells of radius r_0 (radius of earth's surface), having equal volumes with the ice masses, respectively. As the sea covers about three-fourths of the earth's surface the products of these thicknesses by $\frac{4}{3}$ will represent approximately the necessary lowering of the sea level if the water in the ice cap is drawn from the sea. Since the thickness of our assumed ice mass along any radial line at an angular distance β from its axis is by equation (81)

$$h=h_0\left(1-\frac{\sin^n\frac{\beta}{2}}{\sin^n\frac{\beta_0}{2}}\right),$$

the volume of this mass will be expressed by the integral

$$2r_0^2h_0\pi\int_0^{\beta_0}\left(1-\frac{\sin^n\frac{\beta}{2}}{\sin^n\frac{\beta_0}{2}}\right)\sin\beta d\beta=4\frac{n}{n+2}r_0^2h_0\pi\sin^2\frac{2}{\beta_0}.$$

The thickness τ of a spherical shell of radius r_0 and equal volume with the above is given to a sufficient degree of approximation by the relation

$$4r_0^2\tau\pi=4\frac{n}{n+2}r_0^2h_0\pi\sin^2\frac{\beta_0}{2},$$

whence

$$\tau=\frac{n}{n+2}h_0\sin^2\frac{\beta_0}{2}. \qquad (96)$$

From this equation with our working values, $h_0=10{,}000$ feet and $\beta_0=38°$, we find the following values of τ and $\frac{4}{3}\tau$, corresponding to the values of n in the preceding table:

Table showing thicknesses τ of spherical shells of equal volume with assumed ice masses, and equivalent lowering of sea level $\frac{4}{3}\tau$.

n	τ	$\frac{4}{3}\tau$
	Feet.	*Feet.*
1	353	471
2	530	707
3	636	848
4	707	943
5	757	1009
6	795	1060
7	824	1099
8	848	1131
9	867	1156
10	883	1177
∞	1060	1413

48. Although, as seen from the last table, the amount of water necessary to form such an ice cap as we are considering would, if drawn from the sea, cause a decided lowering thereof, yet the mass of ice would be very small compared with the mass of the earth. Thus, for example, the mass of a sheet of uniform thickness 10,000 feet and 38° angular radius is only $\frac{1}{36000}$ part of the earth's mass.

49. To determine the position of the disturbed relative to the undisturbed surface it will be sufficient to compute the elevations of the water at the center of the attracting mass, along its border, and 180° from the center, by means of formulas (90). As the slope of the disturbed surface near the border of the mass is of most importance, we shall compute the elevation of the disturbed surface for a circle of points 1° distant from the border, and thereby deduce the average slope of the disturbed surface within that distance (69 miles) of the border. The separate quantities required in this calculation are given in the following table for the same values of n as those used in the two preceding tables. It will be remembered that $n=\infty$ corresponds to an attracting mass of uniform thickness. The values of S in the last column of the table have all been computed from equation (89), except the one for $n=\infty$, which has been derived from (72).

Table of numerical values of functions defining position of disturbed surface.

n	$\frac{n}{n+1}\sin\frac{\beta_0}{2}$	$\frac{n}{n+2}\sin^2\frac{\beta_0}{2}$	S_1	$2\left(\sin^2\frac{\beta_0}{4}-B\right)$	S
1	0.16278	0.03533	0.06078	0.01796	0.05885
2	.21704	.05300	.09249	.02699	.08946
3	.24418	.06360	.11218	.03243	.10842
4	.26045	.07066	.12568	.03606	.12139
5	.27131	.07571	.13555	.03867	.13085
6	.27906	.07950	.14310	.04063	.13809
7	.28487	.08244	.14908	.04216	.14380
8	.28939	.08480	.15393	.04338	.14843
9	.29301	.08672	.15795	.04438	.15227
10	.29597-	.08833	.16135	.04522	.15550
∞	0.32557	0.10600	0.21111	0.05448	0.19846

For brevity make the following substitutions:

$$N_1=\frac{n}{n+1}\sin\frac{\beta_0}{2}-\frac{n}{n+2}\sin^2\frac{\beta_0}{2},$$

$$N_2=S_1-\frac{n}{n+2}\sin^2\frac{\beta_0}{2},$$

$$N_3=2\left(\sin^2\frac{\beta_0}{4}-B\right)-\frac{n}{n+2}\sin^2\frac{\beta_0}{2},$$

$$N_4=S_1-S.$$

Then from the preceding table we get the following table of values:

n	N_1	N_2	N_3	N_4
1	$+0.12745$	$+0.02545$	-0.01737	$+0.00193$
2	$+.16404$	$+.03949$	$-.02601$	$+.00303$
3	$+.18058$	$+.04858$	$-.03117$	$+.00376$
4	$+.18979$	$+.05502$	$-.03460$	$+.00429$
5	$+.19560$	$+.05984$	$-.03704$	$+.00470$
6	$+.19956$	$+.06360$	$-.03887$	$+.00501$
7	$+.20243$	$+.06664$	$-.04028$	$+.00528$
8	$+.20459$	$+.06913$	$-.04142$	$+.00550$
9	$+.20629$	$+.07123$	$-.04234$	$+.00568$
10	$+.20764$	$+.07302$	$-.04311$	$+.00585$
∞	$+0.21957$	$+0.10511$	-0.05152	$+0.01265$

Now, the factor by which we must multiply N_1, N_2, and N_3 to get the elevation of the disturbed surface above the undisturbed at the center of the mass, along its border, and 180° from its center is

$$3h_0\frac{\rho}{\rho_m}=5454.5 \text{ feet};$$

and one sixty-ninth part of this factor multiplied by N_4 will give the average slope per mile of the disturbed surface within 1° of the border of the ice. Hence we get in the table below the results corresponding to the several values of n. The plus sign indicates elevation and the minus sign depression of the disturbed relative to the undisturbed surface.

Table showing effects in distorting the sea level of ice caps of the same angular radius, 38°, *and same axial thickness,* 10,000 *feet, but of varying external slopes, defined by equation* (81).

n	Position of disturbed relative to undisturbed surface.			Average slope per mile of disturbed surface within 1° of the border of the ice mass.
	At center of ice mas.	Along border of ice mass.	180° from center of ice mass.	
	Feet.	*Feet.*	*Feet.*	*Feet.*
1	+ 695	+139	— 95	0.15
2	+ 895	+215	—142	.24
3	+ 985	+265	—170	.30
4	+1035	+300	—189	.34
5	+1067	+326	—202	.37
6	+1088	+347	—212	.40
7	+1104	+363	—220	.42
8	+1116	+377	—226	.43
9	+1125	+389	—231	.45
10	+1133	+398	—235	.46
∞	+1198	+573	—281	1.00

50. As already explained, the numbers in the second, third, and fourth columns of the last table assign the position, and those in the fifth column the meridian slope of the disturbed surface relative to the undisturbed surface, assuming that the ratio of the density of ice to the earth's mean density is $\frac{2}{11}$, and neglecting the effect of the rearranged free water. If $\frac{2}{11}$ were the correct ratio of the densities, and if the sea covered the whole surface of the earth, and were free to arrange itself in conformity with the attractive forces, formulas (66) and (91) show that the numbers in the second, third, fourth, and fifth columns of the table should be increased by about 15, 18, 26, and 18 per cent., respectively, of their stated amounts. But the ratio $\frac{2}{11}$ is too great by about 8 per cent., so that for this reason the above percentages must be reduced to 7, 10, 18, and 10, respectively. Again, only three-fourths, at most, of the earth's surface is covered with water, so that the reduced percentages must be diminished to three fourths their stated amounts. Moreover, the sea could not penetrate the ice mass in such a manner as to produce the assumed increase in the potential within its border, and hence the reduced percentages must be still further diminished. It is estimated that the actual effect can not be greater than two-thirds that which would follow if the water were unrestricted. Accordingly, $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$ the above reduced percentages, or 3.5, 5, 9, and 5 per cent., respectively, would appear to be liberal allowances for the effect of the free water in exaggerating the deviation of the disturbed from the undisturbed surface at the points designated, over and above the tabular deviations. In view of the smallness of these possible increments to our computed quantities we need give the question of the effect of the rearranged free water no further consideration.

The results in the second column of the table are the heights to which the water would rise at the center of the ice mass if brought within it in any manner, as by a canal, and left free to assume equilibrium. If the amount of free water were sufficient it would rise or fall to the extent indicated in the third and fourth columns, respectively, at points along the border and at the antipodes of the center of the mass.

The slopes given in the fifth column will apply to isolated bodies of free water adjacent to the ice mass, and also to the sea surface in the same vicinity, whether there be sufficient water to rise to the height indicated in the second column or not.[1]

[1] Recent observations of the beaches of bodies of water contiguous with the ice fields of the glacial epoch indicate uniformly that the water surfaces sloped upwards toward the ice. In a careful exploration of the beaches of Lake Agassiz, an extinct lake which lay in the valley of the Red River of the North during the glacial epoch, Professor Upham has found slopes varying from zero to 1.3 feet per mile. (See Bulletin No. 39 of the U. S. Geological Survey, on The Upper Beaches and Deltas of the Glacial Lake Agassiz, by Warren Upham.) Near the south shore of Lake Ontario, in New York, Mr. G. K. Gilbert has observed slopes as great as 5 feet per mile. (See Science, Vol. I, p. 222.)

The maximum possible slope of the disturbed surface would occur at the immediate border of a mass of uniform thickness. This slope is by equation (77) for h=10,000 feet and β=38°, 1.80 feet per mile, and corresponds to a plumb line deflection of 72″. Since it is not probable that the ice cap presented at its border anything like a vertical wall 10,000 feet high, we infer that a mass whose maximum or axial thickness is 10,000 feet would be quite inadequate to produce a slope of 1.8 feet per mile.

Again, since all the results in the table are proportional to the axial thickness of the ice, to produce as great an average slope as 2 feet per mile within 1° (69 miles) of the border of a mass having the more probable slope defined by the index n=6 to n=10, would require an axial thickness in round numbers of 50,000 feet, or 9½ miles. The slope at the immediate border in the extremely improbable case of a uniform thickness of 50,000 feet would be 5×1.8 feet=9 feet per mile.

51. To produce slopes as great as 4 or 5 feet per mile for any distance from its immediate border an ice cap must have great thickness, which implies for any large areal extent a heavy draft on the visible supply of water. The minimum thicknesses of ice masses of varying radial extent which would produce an average slope of 5 feet per mile within 1° (69 miles) of their borders are shown in the following table. These values are computed on the improbable supposition that the masses are of uniform thickness. The volume of each mass is indicated by the equivalent lowering of the sea level, or $\frac{1}{2}\tau$ deduced from equation (96), the value of n being infinite.

Table showing for ice masses of varying radial extent the minimum thicknesses requisite to produce an average slope of 5 feet per mile within 1° of the borders.

Angular radius of mass.	Minimum thickness.	Equivalent lowering of sea level, indicating volume of mass.
°	*Feet.*	*Feet.*
10	69, 400	703
20	52, 500	2, 308
30	52, 600	4, 699
38	50, 000	7, 065

For masses having moderate surface slopes near their perimeters the axial thicknesses must be about twice as great as the minimum values given in the table to produce the same average slope of 5 feet per mile within 1° of the borders. Thus, for the angular extent 38°, and for n=6, say, an average slope of 5 feet per mile would require an axial thickness of about 125,000 feet, or 24 miles. This corresponds to a lowering of the sea of about 2.4 miles, and if the quantity of free water were sufficient, to an elevation of the sea along the border of the mass of about 4,000 feet.

52. Confining further investigation to our working axial thickness, 10,000 feet, angular radius 38°, and ratio of densities $\frac{2}{11}$, we may inquire as to the extent of the variation in sea level at any point of the earth's surface on the supposition of an alternation of glaciation at the poles. For this purpose it is simply necessary to compute the elevation of the disturbed surface at the angular distance α of the point in question from the pole or axis of the ice cap and for the point $180° - \alpha$ and take the difference between the results. Since the maximum effects are of most interest, we shall compute the variations in sea level on the assumption that the ice cap is of uniform thickness. The results, as may be inferred from the second table on p. 66, will not differ materially, except for points near the border of the ice, from the results which would be derived on the assumption of a sloping mass corresponding to the index $n=6$ to $n=10$.

To compute the required integrals we may use formulas (70) and (71). For $\beta=38°$ we find

$\log g_1=9.3493$,	$\log k_1=9.7049$,
$\log g_2=8.7422$,	$\log k_2=8.8079$,
$\log g_3=8.3452$,	$\log k_3=8.3849$,
$\log g_4=8.0708$,	$\log k_4=8.1186$,
$\log g_5=7.8644$,	$\log k_5=7.8904$,
$\log g_6=7.6987$.	$\log k_6=7.7033$.

From the formula

$$w=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\beta}{2}}$$

we compute $\log w$ for $\alpha=10°, 20°, 30°$, etc., to $180°$, and then the values I, or, more conveniently, the values of $I/\pi \sin\frac{\beta}{2}$ required in (64), readily follow. The results are given in tabular form below for circles at intervals of 10° from either pole.

Table showing disturbance of sea level attributable to an ice cap of 38° angular radius and 10,000 feet uniform thickness, and variation in sea level attributable to same mass on the hypothesis of an alternation of glaciation at the two poles.

Angular distance from either pole, or a.	Elevation of disturbed above undisturbed surface, or v_a.	Variation in sea level from epoch of minimum to epoch of maximum glaciation, or $v_a - v_{180-a}$.
°	*Feet.*	*Feet.*
0	+1198	1479
10	+1169	1449
20	+1076	1352
30	+ 893	1163
40	+ 441	702
50	+ 185	434
60	+ 46	279
70	− 43	170
80	− 106	80
90	− 152	0
100	− 186	80
110	− 213	170
120	− 233	279
130	− 249	434
140	− 261	702
150	− 270	1163
160	− 276	1352
170	− 280	1449
180	− 281	1479

A graphical representation of the results in the last two columns of the above table is given in Fig. 6, the section of the disturbed and undisturbed surfaces being so developed that the great circle of the latter appears as a straight line, *AB*. The distances between the two curves representing the disturbed surfaces measured at right angles to the axis *AB* indicate the variation in sea level between the epochs of maximum and minimum glaciation at either pole.

The variation in slope of the sea surface at any point during the interval between the extremes of glaciation will equal the sum of the slopes at that point for the two epochs. The maximum variation is, however, only slightly greater (about 1 per cent.) than the maximum slope already computed, and requires, therefore, no further consideration. That portion of the disturbed surface having the greatest slope is made clearly apparent by the curves in Fig. 6.

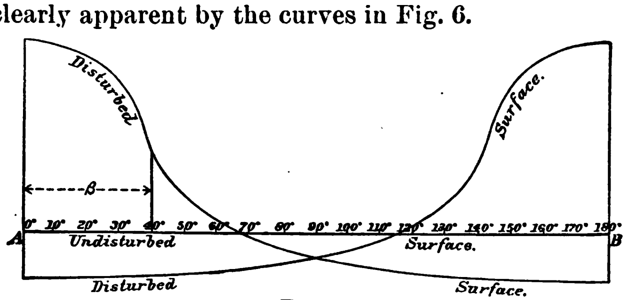

FIG. 6.

Horizontal scale, $\frac{1}{257000000}$; vertical scale, $\frac{36000}{257000000}$.

XV. HISTORICAL NOTE.

53. The effect of the glacial accumulation in disturbing the sea level has been the subject of considerable discussion. The discussion was started apparently by Dr. Croll about 1866; and from the widely differing results quoted by him in his Climate and Time, Chapter XXIV, one might infer that the mathematicians who have attacked the problem are completely at loggerheads. There has been, indeed, some diversity of opinion as to the proper method of treating the question, and the work of one writer at least is quite erroneous; but part of the apparent discrepancy in the results quoted by Dr. Croll is due also to a radical difference in the data used as a basis for calculation.

The mathematicians whose writings on this subject appear to be worthy of especial consideration are Archdeacon Pratt, Mr. D. D. Heath, and Sir William Thomson. Their investigations may be found in the Philosophical Magazine for 1866; Vols. XXXI and XXXII. Those of Archdeacon Pratt are reproduced substantially in his Figure of the Earth, fourth edition, pages 236 to 238, while those of Sir William Thomson are given also in Croll's Climate and Time, Chapter XXIII.

54. *Pratt.*—In the following notice of Pratt's investigations we shall refer to the fourth edition of The Figure of the Earth (published by McMillan & Co., London and New York, 1871). as it is the more recent and formal treatise. Although in many respects a valuable text book, it is marred by some serious errors, one of which we proceed to point out.

Pratt's investigation is based on the following erroneous proposition (see page 212, Figure of the Earth), namely:

To prove that the effect of a mass at the earth's surface, whether above or below, is to make the sea level rise at any place through a space, $\frac{V}{g}$, where V is the potential of the mass for a point on the disturbed sea level, which is in the same vertical line with the place.

By a process of reasoning to which there appears to be no objection he arrives at this equation, namely,

$$r + \text{const} = \frac{V}{g}, \tag{a}$$

in which r is the radius-vector of any point on the disturbed sea surface, V the potential at that point of the disturbing mass, and g is the well known velocity increment due to the earth's attraction. The radius of the undisturbed surface, supposed spherical, being denoted by a, Pratt says:

Let $r=a$ where $V=0$ or the horizontal attraction of the mass first becomes appreciable.

He thus finds in the above equation

$$\text{const} = -a,$$

and hence

$$\text{rise in sea level} = r - a = \frac{V}{g}. \tag{b}$$

But this reasoning is strangely faulty. V is not zero for $r=a$; in fact it is zero only for points infinitely removed from the surface under consideration. He should have reasoned thus: Along the line of intersection of the disturbed and undisturbed surfaces $r=a$. Call the particular value of V along this line V_0. Then we have from (a)

$$\text{const}=-a+\frac{V_0}{g},$$

and hence

$$r-a=\frac{V-V_0}{g}. \qquad (c)$$

This agrees with our formula (6)′, and the constant V_0 is to be determined from the condition that the disturbed and undisturbed surfaces contain equal volumes. Formula (a) gives results which are too great by the constant amount $\frac{V_0}{g}$, i. e., this formula measures heights above a spherical surface $\frac{V_0}{g}$ below the undisturbed surface (see section **5**). V_0, it will be observed, is never of a lower order than V and can not therefore be neglected in comparison with V.

Although Pratt makes use of the correct principle for determining the constant V_0 in his article 199, he ignores this principle altogether in his article 200, referred to above, and again in his article 213.

55. In his calculation Pratt assumes a sheet of ice 7,000 feet thick at the pole to extend over a whole hemisphere, decreasing in thickness, however, as it recedes from the pole in the ratio of the square of the cosine of the polar distance. He does not consider the effect of the rearranged water. His method of determining the potential V is not satisfactory. It consists (see articles 90, 91, and 92, Figure of the Earth) in a species of mechanical quadrature, by which he computes five special values of the attraction of a "hemispherico-spheroidal meniscus," whose thickness varies according to the law stated above. From these five values he derives by the method of indeterminate coefficients a general formula for the attraction of the meniscus, and from this formula by integration he gets a general formula for the potential of the mass. The order of approximation of these formulas is not shown and is not evident. From a test we shall apply it is inferred that the approximation is so rough as to render the formulas worthless.

We may readily derive the proper expression for the elevation of the sea under the conditions assumed by Pratt from our general equation (83). In this equation, if the thickness at the pole be denoted by h_0, we have

$$\varphi(\beta)=h_0\cos^2\beta, \qquad \beta_1=0, \qquad \beta_2=\frac{\pi}{2}.$$

Therefore (83) becomes

$$v''=3\,\frac{h_0\rho}{\pi\rho_m}\left(\int_0^{\frac{\pi}{2}}\frac{dI}{d\beta}\cos^2\beta d\beta-\pi\int_0^{\frac{\pi}{2}}\frac{d\sin^2\frac{\beta}{2}}{d\beta}\cos^2\beta d\beta\right)$$

$$=3\,\frac{h_0\rho}{\pi\rho_m}\int_0^{\frac{\pi}{2}}\frac{dI}{d\beta}\cos^2\beta d\beta-\frac{h_0}{2}\cdot\frac{\rho}{\rho_m}. \qquad (d)$$

The first term in the second member of this equation is $\frac{V}{g}$, and the second term is $\frac{V_0}{g}$ of equation (c) above. Now, for the ratio of the density of ice to the mean density of the earth $\frac{\rho}{\rho_m}$ Pratt uses $\frac{1}{6}$, and hence the constant by which his results should be diminished, if they were what they purport to be, is

$$\frac{h_0}{12}=\frac{7000}{12}\text{ feet}=583\text{ feet.}$$

56. We may test the correctness of Pratt's formula for computing $\frac{V}{g}$ by deriving the elevation of the disturbed surface at the center of his ice sheet. For this point we have by equation (26)

$$I=\pi\sin\frac{\beta}{2},$$

and hence

$$\frac{dI}{d\beta}=\frac{\pi}{2}\cos\frac{\beta}{2}.$$

The integral in (d), therefore, is

$$\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos\frac{\beta}{2}\cos^2\beta d\beta=\pi\left[\left(\frac{1}{2}\right)^{\frac{1}{2}}-\frac{4}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}+\frac{4}{5}\left(\frac{1}{2}\right)^{\frac{5}{2}}\right]=4\pi\,\frac{\sqrt{2}}{15};$$

and (d) becomes

$$v''=\frac{\sqrt{8}}{15}h_0-\frac{1}{12}h_0. \qquad (d)'$$

This assigns the height of the disturbed above the undisturbed surface at the center of the ice mass. Now,

$$\frac{\sqrt{8}}{15}=0.18856,$$

but Pratt's formula (see page 237, Figure of the Earth) gives 0.1189, which is only about 63 per cent. of the true value.[1] By a fortuitous compensation of errors, however, Pratt's formula gives a nearly correct result for the elevation of the sea at the center of the ice cap, the error in the potential for this point being about equal to the constant omitted. His formula gives for this elevation $0.1189h_0=0.1189\times 7000$ feet$=832$ feet. The correct value is $(0.18856-\frac{1}{12})h_0=737$ feet, which is about 13 per cent. of itself smaller than Pratt's value.

57. *Heath.*—The investigations of Heath may be found in numbers CCVIII, CCIX, and CCXIII of the Philosophical Magazine. In number CCVIII he develops the theory of the effect on the sea level of an ice cap of uniform thickness and any angular radial extent, and applies this theory to a numerical example. In number CCIX he corrects a blunder by which he was led in his previous paper to the conclusion that the ice cap would produce a rise of sea level in both hemispheres. His last paper in number CCXIII is chiefly interesting as a review and criticism of the work of Croll, Archdeacon Pratt, and Sir William Thomson.

The method followed by Heath is that of Laplace's functions (spherical harmonics). He takes account of the rearranged water on the supposition that it covers the whole sphere and is free to permeate the ice mass. He considers masses of uniform thickness only, arguing, though not very cogently, that this was the probable form of the ice cap. His mathematical processes are correct in principle, but his formulas defining the position of any point of the disturbed surface are rather uninviting to the computer on account of the slow convergence of the series used. His series is the same as that in the second member of our equation (66), which we have separated into the definite integral and rapidly converging series of the third member of (66).

---

[1] That the first term in the second member of $(d)'$ is $\frac{V}{g}$ for the point in question is easily proved directly. Thus, to terms of the order we neglect, the potential of the mass is

$$V=\rho\int_0^{\frac{\pi}{2}}\frac{2\pi r_0\sin\beta\,.\,h_0\cos^2\beta\,.\,r_0 d\beta}{2r_0\sin\frac{\beta}{2}}$$

$$=2\,r_0 h_0\rho\pi\int_0^{\frac{\pi}{2}}\cos\frac{\beta}{2}\cos^2\beta d\beta$$

$$=4r_0 h\rho\pi\frac{4\sqrt{2}}{15}.$$

Dividing this by $g=\frac{4}{3}r_0\rho_m\pi$ we get

$$\frac{V}{g}=3h_0\frac{\rho}{\rho_m}\,.\,\frac{4\sqrt{2}}{15}=\frac{\sqrt{8}}{15}h_0 \text{ if } \frac{\rho}{\rho_m}=\frac{1}{6}.$$

58. In his numerical example Heath assumes an ice sheet of uniform thickness, 10,000 feet, to extend 30° in every direction from the pole. For the ratio of the density of ice or sea water to the mean density of the earth he uses $\frac{1}{6}$. That is, in the notation of our equation (66) he has

$$h=10{,}000 \text{ feet},$$
$$\beta=30^\circ,$$
$$\frac{\rho}{\rho_m}=\frac{\rho_w}{\rho_m}=\frac{1}{6}.$$

Heath gives the position of the disturbed surface at the two poles and at points 35° distant from either pole. He uses fifty terms of his series. We will verify one of his results, viz, that assigning the elevation of the disturbed surface at the center of the ice cap. With the above data formula (66) becomes

$$v+\Delta v=\frac{5000}{\pi}\left(I-\pi\sin^2\frac{\beta}{2}\right)+2500\sum_{i=1}^{i=\infty}\left(\frac{f_i(\cos\alpha)F_i(\beta)}{4i+1}\right).$$

At the center of the ice cap

$$\alpha=0,$$
$$I=\pi\sin\frac{\beta}{2},$$
$$f_i(\cos\alpha)=1.$$

The numerical values of the first ten terms of

$$\frac{F_i(\beta)}{4i+1}$$

are, in order,

+0.02500,	+0.00058,
+ .01203,	− .00008,
+ .00661,	− .00039,
+ .00358,	− .00047,
+ .00172,	− .00041.

The sum of these is

$$+0.04817.$$

Therefore the above equation becomes

$$v+\Delta v=959 \text{ feet}+120 \text{ feet}=1{,}079 \text{ feet}.$$

For the same result Mr. Heath gives 1,078 feet.

59. In his review Heath first devotes some space to a criticism of the views of Croll and the work of Thomson. Croll had apparently held the notion that the change in sea level due to an ice cap is essentially

equal to the change in position of the center of gravity of the earth and cap. This notion is indeed correct for the particular ideal case considered by Croll and Thomson, but it is in general quite incorrect. Heath points out this fact and concludes that the special case discussed by Croll and Thomson affords us an inadequate conception of the actual problem.

With reference to the investigation of Archdeacon Pratt, which has been published in number CCVIII of the magazine, Heath says:

I must confess, with some diffidence, that it appears to me radically erroneous.

He then proceeds to mention, correctly in the main, but with some reservation, the defects of Pratt's method. On one point, however, he expresses some doubt as to the adequacy of his own process. He is not sure that the order of approximation of his expression for the potential of the ice cap is sufficient, and thinks this may require closer investigation. We have cleared up this point in Article IV.

60. *Thomson.*—The contribution of Sir William Thomson to the discussion of this subject is appended in the form of a note to a paper by Croll, On the Physical Cause of the Submergence and Emergence of the Land During the Glacial Epoch, published in number CCIX of the Philosophical Magazine. The note is brief, but it contains a clear statement of the essential analytical considerations required in the solution of the general problem, and of the decided simplification which results when the ice sheet has the special form assumed for the purpose of discussion by Croll, namely, that of a hemispherical meniscus, whose thickness (or density) varies everywhere as the sine of the latitude. In analogy with the views of Croll, Thomson devises the following ideal conditions, which are interesting as presenting the mechanical features of the problem in their simplest form. He says:

As an assumption leading to a simple calculation, let us suppose the solid earth to rise out of the water in a vast number of small flat-topped islands, each bounded by a perpendicular cliff, and let the proportion of the water area to the whole be equal in all parts. Let all of these islands in one hemisphere be covered with ice, of thickness according to the law assumed by Mr. Croll, that is, varying in simple proportion of the sine of the latitude. Let this ice be removed from the first hemisphere and similarly distributed over the islands in the second.

Thomson gives no analysis, but continues:

By working out according to Mr. Croll's directions, it is easily found that the change in sea level which this will produce will consist in a sinking in the first hemisphere and rising in the second through heights varying according to the same law (that is, simple proportionality to sines of latitudes), and amounting at each pole to

$$\frac{(1-\omega)it}{1-\omega w},$$

where t denotes the thickness of the ice crust at the pole, i the ratio of the density of ice, and w that of sea water to the earth's mean density, and ω the ratio of the area of ocean to the whole surface.

61. We may readily get Thomson's result from equation (66). Thus, the change in position of any point of the sea surface will be, if we

represent the thickness of the ice at any point whose polar distance is β by $h_0 \cos \beta$,

$$\int_0^\pi \frac{d(v+\Delta v)}{d\beta} h_0 \cos \beta d\beta =$$

$$\frac{3}{2} h_0 \frac{\rho}{\rho_m} \sum_{i=1}^{i=\infty} \left(\frac{f_i(\cos \alpha) \int_0^\pi \frac{dF_i(\beta)}{d\beta} \cos \beta d\beta}{1 - \frac{3}{2i+1} \frac{\rho_w}{\rho_m}} \right).$$

But from equation (46)

$$\frac{dF_i(\beta)}{d\beta} = f_i(\cos \beta) \sin \beta;$$

and by the theory of Laplace's functions

$$\int_0^\pi f_i(\cos \beta) \cos \beta \sin \beta d\beta = 0,$$

except when $i=1$. In this case, since $f_i(\cos \beta) = \cos \beta$, the last integral becomes

$$\int_0^\pi \cos^2 \beta \sin \beta d\beta = \frac{2}{3}.$$

Therefore, observing that $f_i(\cos \alpha) = \cos \alpha$, α being the polar distance of any point of the sea surface, we get

$$\int_0^\pi \frac{d(v+\Delta v)}{d\beta} h_0 \cos \beta d\beta = \frac{\frac{\rho}{\rho_m} h_0}{1 - \frac{\rho_w}{\rho_m}} \cos \alpha. \tag{e}$$

Now, since the ratio of the area of the ocean to the whole surface of the earth is assumed in Thomson's problem to be ω, we must replace ρ in (e) by $(1-\omega)\rho$ and ρ_w by $\omega\rho_w$. Making these substitutions, and putting $\alpha=0$, the second member of (e) becomes

$$\frac{(1-\omega) h_0 \frac{\rho}{\rho_m}}{1 - \omega \frac{\rho_w}{\rho_m}}. \tag{f}$$

This is Thomson's result.

Knowing the fact expressed by (e), namely, that the transfer of such a meniscus as we are considering from one hemisphere to the opposite one would change the sea level at any point by an amount proportional to the cosine of the polar distance of that point, we may get the result

(f) by equating the sum of the moments of the transferred ice and water to the moment of the whole earth. In other words, this particular case requires only the amount of shifting of the earth's center of gravity. Calling this amount σ_1, and taking a plane perpendicular to the axis of the meniscus at the disturbed center of gravity of the earth as moment plane, the equation of moments is

$$(1-\omega)\rho\int_0^{\pi} 2\pi r_0 \sin\theta \, . \, r_0 d\theta \, . \, h_0 \cos\theta \, . \, r_0 \cos\theta +$$

$$\omega\rho_w\int_0^{\pi} 2\pi r_0 \sin\theta \, . \, r d\theta \, . \, \sigma_1 \cos\theta \, . \, r \cos\theta = \frac{4}{3}\pi r_0^3 \rho_m \sigma_1 .$$

This gives

$$\sigma_1 = \frac{(1-\omega)h_0\dfrac{\rho}{\rho_m}}{1-\omega\dfrac{\rho_w}{\rho_m}},$$

which is the same as (f).

In his numerical example Thomson takes

$$h_0 = 6{,}000 \text{ feet},$$

$$\omega = \frac{2}{3},$$

$$\frac{\rho}{\rho_m} = \frac{1}{6}, \text{ and } \frac{\rho_w}{\rho_m} = \frac{2}{11}.$$

These data give 379 feet as the change in sea level at the pole during the interval between the epochs of minimum and maximum glaciation. This is the greatest change in sea level that could occur under the assumed conditions.

62. It is to be observed that the numerical results of Pratt (corrected), Heath, and Thomson are not directly comparable with each other nor with the results we have computed in Article XIV, since they are all based on different data. They represent effects due to causes of the same kind, but of widely differing magnitudes. No statement of these results would be intelligible without an accompanying statement of the data on which they rest. To show clearly how widely different writers differ in their data the latter have been collected in a tabular form below. It will be remembered that Pratt took no account of the effect of the rearranged water:[1]

[1] One of the most important contributions to the discussion of the effects of continental glaciers on the sea level has appeared since the manuscript of this paper was placed in the printer's hands, viz: Die Geoiddeformationen der Eiszeit, von Erich von Drygalski, Dr. Phil. W. Pormetter, Berlin, 1887.

Table showing data used by different authors in discussing the problem of glacial submergence.

Author.	Ratio of density of ice to mean density of earth. $\frac{\rho}{\rho_m}$	Ratio of density of sea water to mean density of earth. $\frac{\rho_\omega}{\rho_m}$	Angular radius of ice mass. β_0	Thickness of ice at angular distance β from its axis.
			°	*Feet.*
Pratt	$\frac{1}{6}$	....	90	7,000 $\cos^2\beta$
Heath	$\frac{1}{6}$	$\frac{1}{6}$	30	10,000 uniform.
Thomson	$\frac{1}{6}$	$\frac{2}{11}$	90	6,000 $\cos\beta$.
Woodward	$\frac{1}{6}$	$\frac{2}{11}$.	38	10,000 $\left(1-\frac{\sin^n\frac{1}{2}\beta}{\sin^n\frac{1}{2}\beta_0}\right)$

XVI. VARIATIONS IN SEA LEVEL ATTRIBUTABLE TO CONTINENTAL MASSES.

63. To illustrate the use of the theory developed in determining the disturbance of the sea level attributable to a continental mass we shall consider, in addition to some observations on the general features of the subject, the special case presented by the largest of the continents, namely, Europe and Asia.

In contemplating this problem it is important to distinguish two extreme hypotheses relative to the nature of the earth's crust. On the one hand, we may suppose that a continent is simply a superficial aggregation of matter, which, if removed, would leave a sensibly centrobaric spheroid; in other words, the presence of a continent does not imply that beneath it the earth's crust is any less dense than beneath the ocean. On the other hand, we may suppose that the several radial element prisms of the earth's crust are in a state approximating to hydrostatic equilibrium, and hence, that the mere existence of a continent implies a defect of density in the strata beneath it. According as we proceed from the one hypothesis or the other we shall arrive at widely differing results, which may be regarded, however, as the limits between which the facts lie.

64. Assuming, in accordance with the first hypothesis, that the formation of the continent in question involved a transfer of the earth's center of gravity towards the center of the continent, it will be of interest to compute the position of the disturbed sea surface with respect to spherical surfaces of the same radius and concentric about the original and disturbed centers of gravity, respectively. For this purpose we may use formulas (64) and (79). These are based on the assumption that the attracting mass is of uniform thickness. A more reasonable assumption is, perhaps, that the continents slope up rather rapidly, but not abruptly, from the sea shore, attaining their average height at no

great distance inland; and more accurate knowledge than we now possess might render it desirable to use some form of the more general equation (83), which takes account of variations in thickness of the attracting mass. As we can not expect, however, at present, to represent the actual shape of a continent very closely, and as we shall not attempt to estimate the attraction of the rearranged free water on itself, it will be best to confine attention to masses of uniform thickness, which give effects greater in general than the probable actual effects.

65. For the angular radial extent and the relation of thickness and densities for the continent of Europe and Asia we take the following data substantially as they are given by Helmert in his Geodäsie[1], Part II, pp. 313, 314, viz:

$$\beta = \text{arc of } 38^\circ,$$
$$3h\rho/\rho_m = 4{,}000 \text{ meters}$$
$$= 13{,}124 \text{ feet.}$$

The last expression is arrived at by taking for the average depth of the sea 3,438 meters, and for the average elevation of the continent 440 meters, the density of the continental mass being assumed to be half the earth's mean density, or 2.8, and that of sea water 1. Thus we have a mass 3,438 meters thick, whose effective density is 2.8—1=1.8, and an additional mass 440 meters thick of density 2.8. These two are equivalent, so far as their potential to terms of the order we neglect is concerned, to a single mass of density 1.8 (or $\frac{1}{3}\rho_m$) and 4,120 meters thickness. Hence, in round numbers, the relation above.

The value for the angular radius β of the continent is equivalent to about 2,600 miles, measured along the surface of the earth. It is the same radius assumed in Article XIV for the ice mass. This value is also very nearly that angular extent which a continent of uniform thickness must have to produce the maximum upheaval of water along its border (see section **23**).

The position of the disturbed surface relative to the two spherical surfaces concentric with the original and disturbed centers of gravity, respectively, will be given at intervals of 10° from the center of the continent to the point opposite, or 180° from that center. The requisite values of $I/\pi \sin\frac{\beta}{2}$ for this purpose have been computed from formulas (70) and (71), using the logarithms of g_1, g_2, etc., k_1, k_2, etc., given on page 69. To determine the slope of the disturbed surface with respect to the spherical surfaces of reference (or the deflections of the plumb line) at intervals of 10° from the center of the continent, use has been made of equations (73) and (80), the differential coefficients $dI/d\alpha$ being computed from (74) and (75). Omitting the details of the computation, the nature of which is readily apparent from the

[1] For full title see page 86.

equations referred to, the results are embodied in the following table. The first column of the table gives angular distances increasing by increments of 10° along a great circle from the center of the continent. The second and third columns give for each angular distance the elevation or depression of the disturbed surface relatively to the spherical surfaces concentric with the undisturbed and disturbed centers of gravity of the earth, respectively, elevations being indicated by the plus sign and depressions by the minus sign. The fourth and fifth columns give for each angular distance the deflections of the plumb line or inclinations of the disturbed surface to the spherical surfaces of reference. The signs of these deflections are minus or plus according as the angle between the plumb line or normal at any point of the disturbed surface and the axis of the continent is greater or less than the angle between the radius vector of the same point and the axis of the continent (see Fig. 7).

Table showing the superior limiting effects in disturbing the sea level attributable to the continent of Europe and Asia.

Angular distance from center of continent.	Elevation or depression of disturbed surface with respect to—		Inclination of disturbed surface with reference to—	
	Spherical surface concentric with undisturbed center of gravity of earth.	Spherical surface concentric with disturbed center of gravity of earth.	Spherical surface concentric with undisturbed center of gravity of earth.	Spherical surface concentric with disturbed center of gravity of earth.
°	*Feet.*	*Feet.*	″	″
00	+2,881	+1,637	− 0.0	− 0.0
10	+2,812	+1,587	− 7.7	− 5.6
20	+2,588	+1,419	−17.7	−13.5
30	+2,148	+1,072	−33.9	−27.7
40	+1,099	+ 146	−51.3	−43.5
50	+ 444	− 356	−24.2	−14.8
60	+ 111	− 511	−14.7	− 4.1
70	− 104	− 529	−10.0	+ 1.6
80	− 235	− 471	− 7.2	+ 4.9
90	− 365	− 365	− 5.4	+ 6.9
100	− 448	− 232	− 4.1	+ 8.0
110	− 512	− 87	− 3.2	+ 8.4
120	− 561	+ 61	− 2.4	+ 8.2
130	− 600	+ 200	− 1.9	+ 7.5
140	− 629	+ 324	− 1.4	+ 6.5
150	− 650	+ 426	− 1.0	+ 5.1
160	− 665	+ 504	− 0.7	+ 3.5
170	− 673	+ 552	− 0.3	+ 1.8
180	− 676	+ 568	− 0.0	+ 0.0

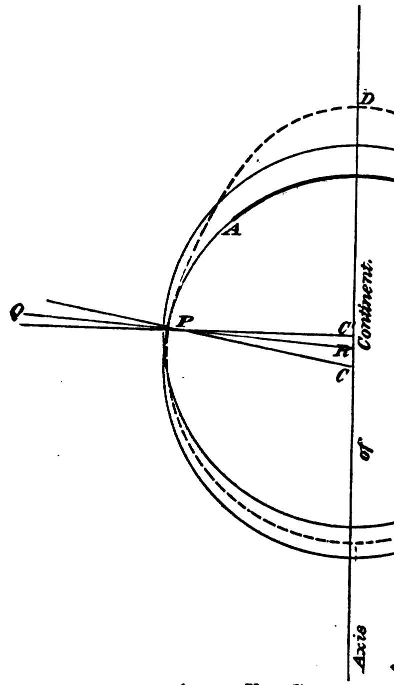

FIG. 7.

66. The relations of the disturbed surface to the spherical surfaces of reference are shown in Fig. 7. The full-line circles in this diagram represent great circles of the spheres of reference on a scale of $\frac{1}{250,000,000}$. The position of the continent is indicated by the arc AB of the circle, whose center C is the undisturbed center of gravity of the earth. The disturbed center of gravity, or the center of the other circle, is at C'. The position of the disturbed surface is indicated by the dotted line. The radial distances of this line from the circles of reference and the distance between the centers C and C' are exaggerated 2,500 times. QPR represents a normal to the disturbed surface at P.

67. For the elevation of the disturbed surface at the immediate border of the continent we find from the second of equations (67) and from (79) 1,380 feet or 400 feet, according as the sphere of reference is concentric with the undisturbed or with the disturbed center of gravity of the earth.

68. It should be remarked in this connection that the results given in the preceding table, and, indeed, all those in this section pertaining to the continent of Europe and Asia, are such as would exist under the assumed conditions if there was no other continent. The complete problem requires the determination of the resultant action of all the continents at any point of the sea surface. For a method of determining this resultant, when the components due to the several continents are known, the reader may be referred to Helmert's Geodäsie.

69. To determine the deflection of the plumb line at the level of the sea along the border of the continent, we observe that according to our assumed data the portion of the continental mass lying below the sea level is about 11,000 feet thick and has an effective density of 1.8, while the portion above the sea level is about 1,400 feet thick and has an effective density of 2.8. We may compute the deflection of the plumb line due to each of these portions by means of equation (77) and add the results to get the total deflection. For the lower mass we have for use in (77)

$$h=11{,}000 \text{ feet}, \qquad \rho=1.8, \qquad v=11{,}000 \text{ feet};$$

and for the upper mass

$$h=1{,}400 \text{ feet}, \qquad \rho=2.8, \qquad v=0 \text{ feet}.$$

Hence the deflections due to the upper and lower masses are 138″ and 34″, respectively, and their sum is 172″. This deflection is relative to a radius drawn to the undisturbed center of gravity of the earth. The deflection relative to a radius drawn to the disturbed center of gravity is by equation (80) 8″ less, or 164″.

70. Under the conditions of the second hypothesis, which supposes the several radial element prisms of the earth's crust in a state bordering on hydrostatic equilibrium, it is evident that the disturbances of the sea level attributable to a continental mass must be of a low order. They must, in fact, be confined to terms of no higher order than those which have been neglected in our equations defining the position of the disturbed sea surface. The precise evaluation of these terms would offer difficulties practically insuperable in all cases, except those which present the simplest arrangement of densities in the element prisms of the earth's crust. We may form a sufficiently definite idea of their smallness, however, by considering the ideal question of the effect on the sea level of the radial transfer of a stratum of the earth's crust from some position below to some position above the sea surface.

Let the stratum considered be a portion of a spherical shell; let its border be circular and of angular radius β. The effect on the sea level will obviously be greatest at the axis of the stratum. Hence, we only need to derive the change in potential at the sea level and at this axis due to the transferred stratum. Suppose the stratum raised to a height h_1 above sea level. Let its uniform thickness in this position be t_1 and its uniform density ρ_1. Then, r being the radius vector of any element of the mass and r' the radius vector of the sea surface, the potential of the stratum at its axis and at sea level will be

$$2\rho_1\pi\int_{r'+h_1}^{r'+h_1+t_1} r^2dr\int_0^{\beta}\frac{\sin\theta d\theta}{\sqrt{r^2+r'^2-2rr'\cos\theta}}=$$

$$2\rho_1\pi\int_{r'+h_1}^{r'+h_1+t_1}\left[\sqrt{r^2+r'^2-rr'\cos\beta}-(r-r')\right]\frac{rdr}{r'}. \qquad (167)$$

This to terms of the first order inclusive is

$$4r'_{\cdot}t_1\rho_1\pi \sin\frac{\beta}{2}\left(1+\frac{3t_1+6h_1}{4r'}-\frac{t_1+2h_1}{4r'\sin\frac{\beta}{2}}\right). \tag{97}$$

If we suppose the stratum in its original position is below the sea surface a distance h_2, and that its uniform thickness and density in this position are t_2 and ρ_2, respectively, the resulting potential at the same point of the sea surface is

$$4r'\rho_2t_2\pi \sin\frac{\beta}{2}\left(1-\frac{3t_2+6h_2}{4r'}-\frac{t_2+2h_2}{4r'\sin\frac{\beta}{2}}\right). \tag{98}$$

Since $\rho_1t_1=\rho_2t_2$; i. e., since the mass of the stratum in the two positions is the same, though its thickness and density may vary, the difference of (97) and (98) is

$$4r'\rho_1t_1\pi \sin\frac{\beta}{2}\left(\frac{3t_1+3t_2+6h_1+6h_2}{4r'}-\frac{t_1-t_2+2h_1-2h_2}{4r'\sin\frac{\beta}{2}}\right). \tag{99}$$

This expression shows that a stratum of the earth's crust might be transferred through a considerable distance radially without materially affecting the sea level.

Thus, for example, suppose

$$h_1=h_2=10{,}000 \text{ feet}, \qquad t_1=t_2=5{,}000 \text{ feet}.$$

Then the fraction in the parenthesis of (99) becomes $\frac{1}{560}$. In other words, the potential at the sea surface under the center of the stratum would be only $\frac{1}{560}$ as great as that due to an uncompensated external stratum or internal vacuity of equal effective mass.

In the case of the continent of Europe and Asia we have (section **69**)

$$h_1=0, \qquad t_1=1{,}400 \text{ feet}, \qquad \rho_1=2.8.$$

These in (97) give

$$4r'\pi \sin\frac{\beta}{2}\,1400\times2.8\left(1+\frac{1}{20000}-\frac{1}{60000\sin\frac{\beta}{2}}\right). \tag{a}$$

Likewise for use in (98) we have

$$h_2=0, \qquad t_2=11{,}000 \text{ feet}, \qquad \rho_2=1.8.$$

These give

$$4r'\pi \sin\frac{\beta}{2}\,11000\times1.8\left(1-\frac{1}{2550}-\frac{1}{7640\sin\frac{\beta}{2}}\right). \tag{b}$$

The sum of (a) and (b) reduced to unit density is

$$4r'\pi \sin\frac{\beta}{2}\,23720\left(1-\frac{1}{3000}-\frac{1}{9000\sin\frac{\beta}{2}}\right). \qquad (c)$$

This is the potential to terms of the first order, inclusive of the whole continent at its center and at sea level, on the supposition that there is no defect in density of the underlying strata. But if we suppose a defect in density of 0.237 uniformly distributed through a depth of 100.000 feet below the sea level, we must diminish the result (c) by

$$4r'\pi\frac{\beta}{2}\,23720\left(1-\frac{1}{280}-\frac{1}{840\sin\frac{\beta}{2}}\right).$$

The remainder is

$$4r'\pi\frac{\beta}{2}\,23720\left(\frac{1}{309}+\frac{1}{900\sin\frac{\beta}{2}}\right),$$

or about $\frac{1}{300}$ of the potential due to the uncompensated mass. Under these conditions of compensation, therefore, the elevation of the sea at the center of the continent of Europe and Asia would be only about 10 feet, whereas the first hypothesis would require an elevation of about 2,900 feet (see section **65**.)

71. Notwithstanding the feeble effect a continental mass, whose radial elements are in a state approximating to hydrostatic equilibrium, would have in elevating or depressing the sea surface, it is conceivable that a considerable deflection of the plumb line might be produced by such a mass along its border. If, for example, we suppose the visible mass of the continent of Europe and Asia to be compensated by a defect in density of 0.18 uniformly distributed through a depth of 135,000 feet below sea level, the deflection of the plumb line along the border of this continent would be diminished by 117″, leaving still a deflection (section **69**) of 172″ − 117″ = 55″ relative to the undisturbed sphere of reference. This deflection, 55″, it should be observed, is the maximum possible value under the assumed conditions. If the degree of compensation assumed actually exists, it is probable that the real maximum deflection of the plumb line is much less than 55″, since our calculation premises a vertical coast wall for the continent, whereas it presents for the most part only a moderately steep slope along the sea shore.

XVII. LIST OF AUTHORS CONSULTED.

72. The authors whose works have been specially consulted in the preparation of the preceding pages, and the full titles of their works, dates of publication, etc., are named below:

Bruns, Dr. Heinrich. Die Figur der Erde. Publication des königl. preussischen geodätischen Institutes. Berlin, 1878.

Clarke, Col. A. R. Geodesy. Oxford, 1880.

Fischer, Dr. Philipp. Untersuchungen über die Gestalt der Erde. Darmstadt, 1868.

Helmert, Dr. F. R. Die mathematischen und physikalischen Theorieen der höheren Geodäsie. II Teil. Leipzig, 1884.

Laplace. Mécanique Céleste. Tome 3. Paris, 1878.

Pratt, John H. A Treatise on Attractions, Laplace's Functions, and the Figure of the Earth. Fourth edition. London and New York, 1871.

Stokes, G. G. Mathematical and Physical Papers, Vol. II, especially the paper On the Variation of Gravity at the Surface of the Earth. Cambridge University Press, 1883.

Thomson and Tait. Treatise on Natural Philosophy. Vol. I, Part II. Cambridge, 1883.

INDEX.

DATE DUE

GAYLORD PRINTED IN U.S.A.